

Amalgamation, Characterizing Cardinals and Locally Finite Abstract Elementary Classes*

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Abstract

We introduce the concept of a locally finite abstract elementary class and develop the theory of excellence for such classes. From this we find a family of complete $L_{\omega_1, \omega}$ sentences ϕ^r such that a) ϕ^r is r -excellent but b) ϕ^r homogeneously characterizes \aleph_r (improving results of Hjorth [8] and Laskowski-Shelah [9] and answering a question of [14]) while c) the ϕ_r provide the first example of an abstract elementary class where the spectrum of cardinals on which amalgamation holds contains more than one interval.

1 Introduction

Amalgamation, finding a model M_2 in a given class \mathbf{K} into which each of two extensions of a model $M \in \mathbf{K}$ can be embedded, has been a theme in model theory in the almost 60 years since the work of Jónsson and Fraïssé. An easy application of compactness shows that amalgamation holds for every triple of models of a complete first order theory. For an $L_{\omega_1, \omega}$ -sentence, ϕ , the situation is much different; there can be a bound on the cardinality of models of ϕ and whether the amalgamation property holds can depend on the cardinality of the particular models. Shelah generalized the Jónsson context to that of an abstract elementary class by providing axioms governing the notion of *strong substructure*. He introduced the notion of n -dimensional amalgamation in a cardinal λ and used it to prove that excellence (r -dimensional amalgamation in \aleph_0 for every $r < \omega$) implies ϕ has arbitrarily large models. We show the requirement on all $r < \omega$ is necessary¹ by constructing for each r a sentence ϕ^r which satisfies r -dimensional amalgamation in \aleph_0 but has no model in \aleph_{r+1} .

In [8], Hjorth finds, by an inductive procedure, for each $\alpha < \omega_1$, a complete sentence $\phi_\alpha \in L_{\omega_1, \omega}$ which characterizes \aleph_α (ϕ_α has a model of that cardinality but no larger model). This procedure was non-uniform in the sense that he showed one of two sentences worked at each ϕ_α ; it is conjectured [15] that it may be impossible to decide in ZFC which sentence works. In this note, we show a simplification of the

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¹Hart-Shelah [7] provided an earlier example showing there are ϕ^r categorical up to \aleph_r but then losing categoricity. Those examples have arbitrarily large models and satisfy amalgamation in all cardinals [2].

Laskowski-Shelah example (see [9, 4]) gives a family of $L_{\omega_1, \omega}$ -sentences ϕ^r , which homogeneously (see Definition 4.21) characterize \aleph_r for $r < \omega$.

The (*finite*) *amalgamation spectrum* of an abstract elementary class \mathbf{K} is the set $X_{\mathbf{K}}$ of $n < \omega$ such that \mathbf{K} has a model in \aleph_n and satisfies amalgamation² in \aleph_n . There are only a few known spectra.

For complete sentences of $L_{\omega_1, \omega}$ there are many examples where the spectrum is either \emptyset or ω . The first example of a complete sentence failing amalgamation in \aleph_0 is due to Kueker [10].

Well-orderings of order type at most \aleph_r under end extension have spectra $\{0, 1, \dots, r\}$. Note however that the Löwenheim number is the same as the size of the maximal model and these classes are not $L_{\omega_1, \omega}$ axiomatizable. An incomplete sentence with finite amalgamation spectrum $\omega - \{0\}$ is given in [5].

We show in ZFC that for each $r \geq 1$, $\{0, 1, \dots, r-2\} \cup \{r\}$ is an amalgamation spectrum by a *complete* $L_{\omega_1, \omega}$ sentence ϕ^r thus making the strongest requirement on the way the class is defined while getting the most unusual spectrum. More precisely, the $L_{\omega_1, \omega}$ -sentence ϕ^r has disjoint 2-amalgamation up to and including \aleph_{r-2} ; disjoint amalgamation and even amalgamation fail in \aleph_{r-1} but hold (trivially) in \aleph_r ; there is no model in \aleph_{r+1} . The previous best result in this direction [5] had disjoint amalgamation up to \aleph_{k-3} , no model in \beth_k and the sentence was not complete.

We have formulated these results in terms of abstract elementary classes for further applications. But the reader can follow the proof thinking only of the concrete results being proved.

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2 Fraïssé Constructions and locally finite AEC

{two}

In [9] Laskowski-Shelah constructed by a Fraïssé construction, which is easily seen to satisfy disjoint amalgamation in the finite, a complete sentence ϕ_{LS} in $L_{\omega_1, \omega}$ such that every model in \aleph_1 is maximal and so characterizes \aleph_1 . The idea was to exhibit a countable family of binary functions that give rise to a locally finite closure relation, and then to note that any locally finite closure relation on a set of size \aleph_2 must have an independent subset of size three. They also asserted that they had a similar construction to characterize all cardinals up to \aleph_ω , but there was a mistake in the proof.³

Here, we use the induction principles introduced by Shelah in the development of excellent classes to uniformly construct families of functions that do give rise to locally finite closure relations that characterize \aleph_r for any integer $r \geq 1$.

For a fixed $r \geq 1$, let τ_r be the (countable) vocabulary consisting of countably many $(r+1)$ -ary functions f_n and countably many $(r+1)$ -ary relations R_n . We consider the class \mathbf{K}^r of finite τ_r -structures (including the empty structure) that satisfy the following three sentences of $L_{\omega_1, \omega}$:

- The relations $\{R_n : n \in \omega\}$ partition the $(r+1)$ -tuples;
- For every $(r+1)$ -tuple $\mathbf{a} = (a_0, \dots, a_r)$, if $R_n(\mathbf{a})$ holds, then $f_m(\mathbf{a}) = a_0$ for every $m \geq n$;
- There is no independent subset of size $r+2$.

The third condition refers to the closure relation on a τ_r -structure M defined by iteratively applying the functions $\{f_n\}$, i.e., for every subset $A \subseteq M$, $\text{cl}(A)$ is the smallest substructure of M containing A . A set B is *independent* if, for every $b \in B$, $b \notin \text{cl}(B \setminus \{b\})$.

²For the precise definitions of amalgamation see Definition 4.21 and Remark 4.4. We say amalgamation holds in κ in the trivial special case when all models in κ are maximal. We say amalgamation fails in κ if there are no models to amalgamate.

³Specifically, the closure relation defined in Lemma 0.7 of [9] is not locally finite.

We develop two classes of models from the class \mathbf{K}^r of finite structures. The first is defined directly from \mathbf{K}^r , while the second can be viewed interchangeably as the class of atomic models of an associated first order theory or the models of a *complete* sentence in $L_{\omega_1, \omega}$.

Definition 2.1. $\hat{\mathbf{K}}^r$ denotes the class of all τ_r -structures M (including the empty structure) with the property that every finite subset $A \subseteq |M|$ is contained in a (finite) substructure $N \in \mathbf{K}^r$ of M .

We make one change from the standard ([13] or [1]) definition of abstract elementary class by replacing the usual notion of Löwenheim-Skolem number by the following.

Definition 2.2. A class \mathbf{K} of structures and a substructure relation $\prec_{\mathbf{K}}$ is a locally finite abstract elementary class if it satisfies the normal axioms for an AEC except the usual Löwenheim-Skolem condition is replaced by: If $M \in \mathbf{K}$ and $A \subset M$ of M is finite, there is a finite $N \in \mathbf{K}$ with $A \subset N \prec_{\mathbf{K}} M$ (read N is a strong substructure).

{lfaec}

Clearly the class $\hat{\mathbf{K}}^r$, equipped with the ordinary substructure relation as $\prec_{\mathbf{K}}$, is a locally finite abstract elementary class.

Definition 2.3. Let $(\mathbf{K}, \prec_{\mathbf{K}})$ be a (locally finite) abstract elementary class

1. The model M is finitely \mathbf{K} -homogeneous or rich if for all finite $A, B \in \mathbf{K}$, $A \prec_{\mathbf{K}} M$, $A \prec_{\mathbf{K}} B \in \mathbf{K}$ implies there exists $B' \prec_{\mathbf{K}} M$ such that $B \cong_A B'$.
2. The model M is generic if M is rich and M is an increasing union of finite substructures that are strong in M .

With Corollary 4.2 we will see that the class $\hat{\mathbf{K}}^r$ has a (unique) countable generic model M^r . Let ϕ^r denote the Scott sentence of M^r and let T^r denote the first order theory of M^r . Moreover, M^r will be seen to be an atomic model of T^r and a τ_r -structure N is an atomic model elementarily equivalent to M^r if and only if N realizes the $L_{\omega_1, \omega}$ -sentence ϕ^r (See chapter 6 of [1]). Consequently we write:

Notation 2.4. The class \mathbf{At}^r consists of all atomic models of T^r (i.e. all models of ϕ^r). When equipped with elementary substructure as $\prec_{\mathbf{K}}$, $(\mathbf{At}^r, \prec_{\mathbf{K}})$ is an abstract elementary class with Löwenheim-Skolem number \aleph_0 .

It is easy to check that \mathbf{K}^r satisfies the amalgamation property, indeed the disjoint amalgamation property. See Definition 3.2 for precise definitions, [4] for the general background, the case $r = 1$ of Lemma 4.1 for a detailed argument and Remark 4.4 for various notions of amalgamation.

Following [8] and [4] we vary from Fraïssé's original construction by a) allowing function symbols in the language b) not insisting the class is closed under substructure but c) guaranteeing local finiteness by axiomatizing with the Scott sentence. Disjoint amalgamation greatly simplifies the arguments that follow. We establish below various levels of amalgamation for $\hat{\mathbf{K}}^r$ and then transfer them to \mathbf{At}^r .

3 Excellence and disjoint amalgamation

The key to this investigation is the study of excellence in locally finite abstract elementary classes. In this section we develop general methods for building models in larger cardinals. Our main construction in Section 4 will take the \mathbf{K} of this section as $\hat{\mathbf{K}}^r$. Substructure is the natural substructure relation on $\hat{\mathbf{K}}^r$ since

{excsec}

the axioms are all universal. For \mathbf{At}^r , first order elementary submodel is the natural submodel relation. It is however easy to see that here this is the same as $L_{\infty, \omega}$ -submodel.

Excellence was first formulated [11, 12] in an ω -stable context. Shelah develops there a substantial apparatus to define ‘independence’ and excellence concerns ‘independent systems’. The applications here require much less machinery because we are able to exploit disjoint amalgamation. ‘Independent amalgamation’ in the usual sense implies disjoint amalgamation (but not strong disjoint amalgamation) as defined below. In using our next definition of $(\langle \lambda, k \rangle)$ -system, we take advantage of the fact that one can always rearrange the data of an amalgamation problem so the domains intersect exactly where required. In general, of course, the embeddings then force identifications.

Definition 3.1. *A set of \mathbf{K} -structures $\overline{N} = \langle N_u : u \subsetneq k \rangle$ is a $(\langle \lambda, k \rangle)$ -system for \mathbf{K} if for $u, v \subsetneq k$:*

1. $\|N_u\| < \lambda$;
2. if $u \subset v$ then $N_u \subset N_v$;
3. $N_u \cap N_v = N_{u \cap v}$.

Definition 3.2. *We say that \mathbf{K} has disjoint $(\langle \lambda, k \rangle)$ -amalgamation if*

{genapdef}

1. $k = 0$ and there is $M \in \mathbf{K}$ with $\|M\| = \mu$ for all $\mu < \lambda$.
2. $k = 1$ and for all $\mu < \lambda$, each $M \in \mathbf{K}$ with $\|M\| = \mu$ has a proper extension.
3. $k \geq 2$ and for any $(\langle \lambda, k \rangle)$ -system \overline{N} there is a model $M \in \mathbf{K}$ such that for every $u \subsetneq k$: N_u is a substructure of M .

We now use the standard idea of excellence: going up a cardinal loses one dimension of the amalgamation. But the argument is much simpler than in [12, 1] or even [5]. (There is an exposition in [3].) We give the full proof to show the core idea of the method in transparent way. For this induction step we need the notion of a filtration of a $(\langle \lambda, n \rangle)$ -system. We use standard notation: $\mathcal{P}^-(n)$ denotes the subsets of n that have at most $n - 1$ elements.

Definition 3.3. *Suppose $\mathcal{S} = \langle M_s : s \in \mathcal{P}^-(n) \rangle$ is a $(\langle \lambda, n \rangle)$ -system. A filtration of \mathcal{S} is a system $\langle \mathcal{S}^\alpha : \alpha < \lambda \rangle$ where each $\mathcal{S}^\alpha = \langle M_s^\alpha : s \in \mathcal{P}^-(n) \rangle$ such that:*

1. each $\|M_s^\alpha\| = |\alpha|$ if α is infinite and finite if α is finite;
2. for each s in $\mathcal{P}^-(n)$, $\{M_s^\alpha : \alpha < \lambda\}$ is a filtration of M_s ;
3. for each α , \mathcal{S}^α is an $(\langle \alpha^*, n \rangle)$ -system where α^* is \aleph_0 if α is finite and $|\alpha|^+$ if α is infinite.

Claim 3.4. *For any $s < \omega$, if $(\mathbf{K}, \prec_{\mathbf{K}})$ has the disjoint $(\langle \lambda, s + 1 \rangle)$ -amalgamation property, then it has the disjoint $(\langle \lambda^+, s \rangle)$ -amalgamation property.*

{exc}

Proof. We first note that the case $s = 0$ is easy. We get a model in λ by taking the union of an increasing chain of smaller models by $(\lambda, 1)$ -amalgamation. For $s = 1$, filter a model M of cardinality λ by $\langle M_i : i < \lambda \rangle$; take an extension N_0 of M_0 ; amalgamate N_0 with M_1 over M_0 to get N_1 ; by disjoint amalgamation N_1 is a proper extension of M_1 ; iterate.

Now let $s \geq 2$ and let $\mathcal{S} = \overline{N} = \langle N_t : t \in \mathcal{P}^-(s) \rangle$ be a $(\langle \lambda, s \rangle)$ -system. Choose a filtration $\langle \mathcal{S}^\alpha : \alpha < \lambda \rangle = \langle N_t^\alpha : t \in \mathcal{P}^-(s), \alpha < \lambda \rangle$ of \mathcal{S} . We will define a sequence $\langle N^\alpha : \alpha < \lambda \rangle$ of models of

cardinality $< \lambda$ whose union is the required amalgam of \mathcal{S} and auxiliary $(< \lambda, s + 1)$ systems \hat{S}^α for $\alpha > 0$. Let N^0 be the amalgam of the system \mathcal{S}^0 . If we have amalgamated \hat{S}^α to obtain a model N^α , we define $\hat{S}^{\alpha+1}$ to have models $\langle N_t^\alpha : t \in \mathcal{P}^-(s) \rangle \cup \{N^\alpha\} \cup \langle N_t^{\alpha+1} : t \in \mathcal{P}^-(s) \rangle$ with the embeddings given by the filtration from N_t^α to $N_t^{\alpha+1}$ and by the given maps within \mathcal{S}^α and $\mathcal{S}^{\alpha+1}$. $\square_{3.4}$

In the case of strong disjoint amalgamation N^α has the same domain as \mathcal{S}^α but has had further functions defined to make it a member of $\hat{\mathbf{K}}$. A vastly more complicated version of this result is expounded in chapters 20 and 21 of [1]; Corollary 21.7 most closely approximates the construction here.

The key points (proved in Theorem 4.3) for our application to $\hat{\mathbf{K}}^r$ and its ‘existential completion’, \mathbf{At}^r , are that disjoint 2-amalgamation in \aleph_s for $\hat{\mathbf{K}}^r$ implies that any member of $\hat{\mathbf{K}}^r$ extends to a model in \mathbf{At}^r and even that we have a disjoint (not strongly disjoint) amalgamation in \aleph_s for \mathbf{At}^r for $s \leq r - 2$. These imply that for $s \leq r - 2$ every model of \mathbf{At}^r in \aleph_{s+1} has a proper extension and so T^r has an atomic model in \aleph_{s+2} , thus in particular in \aleph_r .

4 Characterizing \aleph_n and the amalgamation spectra

{results}

We begin by studying \mathbf{K}^r , the finite models in $\hat{\mathbf{K}}^r$. Then we use the general properties established in Section 3 to construct models in certain cardinals. We then note the combinatorial Fact 4.5 and deduce from it the negative results: non-existence and failure of amalgamation in certain cardinals.

We will verify here a slightly stronger notion: *strong disjoint amalgamation*: replace in Definition 3.2.3 ‘ N_u is a substructure of M ’, by ‘the universe of M is $\bigcup_{u \subseteq k} |N_u|$ ’.

{genap}

Theorem 4.1. *For each $r \geq 1$, \mathbf{K}^r has strong disjoint $(< \aleph_0, r + 1)$ -amalgamation. Further, \mathbf{K}^r does not have disjoint $(< \aleph_0, r + 2)$ -amalgamation.*

Proof. Fix a $(< \aleph_0, r + 1)$ system $\langle N_u : u \in \mathcal{P}^-(r + 1) \rangle$ from \mathbf{K}^r . Let $D = \bigcup_{u \in \mathcal{P}^-(r+1)} |N_u|$ and fix an enumeration $\langle d_i : i < t \rangle$ of D , where $t = |D|$. Call an $(r + 1)$ -tuple $\mathbf{b} \in D^{r+1}$ *unspecified* if $\mathbf{b} \notin N_u^{r+1}$ for any $u \subseteq (r + 1)$. If our amalgam M is to have universe D with each N_u a substructure of M , we need only define the functions f_n and the relations R_n on unspecified tuples.

For every unspecified $\mathbf{b} = (b_0, \dots, b_r)$, put $R_t^M(\mathbf{b})$ and define

$$f_i^M(\mathbf{b}) = \begin{cases} d_i & \text{if } i < t \\ b_0 & \text{otherwise} \end{cases}$$

To see that $M \in \mathbf{K}^r$, we show that there are no independent subsets of size $r + 2$. Choose any $B \subseteq M$ of size $r + 2$. Let $\{u_j : j < r + 1\}$ list the subsets of $(r + 1)$ of size r . If $B \subseteq N_{u_j}$ for any j , then B is not independent since $N_{u_j} \in \mathbf{K}^r$. However, if B is not contained in any N_{u_j} , then there is an unspecified $\mathbf{b} \in B^{r+1}$. In this case, B is contained in the closure of \mathbf{b} , so again B is not independent.

For the second sentence, choose a $(< \aleph_0, r + 2)$ -system $\langle N_u : u \in \mathcal{P}^-(r + 2) \rangle$ from \mathbf{K}^r with the property that for every $j \in r + 2$, there is an element $b_j \in |N_{\{j\}}| \setminus |N_\emptyset|$. Suppose $M \in \mathbf{K}^r$ contains $\bigcup\{|N_u| : u \in \mathcal{P}^-(r + 2)\}$. We claim that $B = \{b_j : j \in r + 2\}$ is independent in M , contradicting the definition of \mathbf{K}^r . To see this, let $B_k = B \setminus \{b_k\}$ for any $k \in r + 2$. Then $B_k \subseteq |N_{(r+2) \setminus \{k\}}|$ and it follows that in M , $\text{cl}(B_k) \subseteq |N_{(r+2) \setminus \{k\}}|$. In particular, $\text{cl}(B_k)$ does not contain b_k . $\square_{4.1}$

{Fraisse}

Corollary 4.2. *The class \mathbf{K}^r has a Fraïssé limit M^r . It is countable, generic, and is an element of $\hat{\mathbf{K}}^r$. It is the unique countable atomic model of its first order theory T^r . Moreover, a τ_r -structure N is in \mathbf{At}^r if and only if (1) every finite subset $A \subseteq N$ is contained in a finite substructure of N that is an element of \mathbf{K}^r and (2) N is \mathbf{K}^r -rich.*

Proof. The existence of a countable, rich structure $M^r \in \hat{\mathbf{K}}^r$ follows from the fact that \mathbf{K}^r satisfies disjoint amalgamation, and this fact follows immediately from Theorem 4.1. To see that M^r is atomic, note that for any finite substructures $B, C \subseteq M^r$, both containing a substructure A , any isomorphism from B to C fixing A pointwise extends to an automorphism of M^r . In particular, $\text{tp}(B/A)$ is isolated by a formula describing which $R_n(\mathbf{b})$ holds, as well as the values of $f_i(\mathbf{b})$ for every $i < n$, for every $(r + 1)$ -element sequence from B . $\square_{4.2}$

The following formula uses \aleph_{-1} where $< \aleph_0$ is used above.

Theorem 4.3. *For each r with $1 \leq r < \omega$ and $-1 \leq s \leq r$, \mathbf{K}^r has strong disjoint $(r - s)$ -amalgamation in \aleph_s for $s \leq r$. This implies there are atomic models of T^r of cardinality \aleph_s for $s \leq \aleph_r$.* {getmod}

Proof. The first sentence is just a calculation from Theorem 4.1 and Claim 3.4. In particular, just reading off from Definition 3.2 we get models up to \aleph_r for $\hat{\mathbf{K}}^r$.

For the second sentence, note first that T^r has an atomic model in \aleph_s for $s \leq r - 2$ because the strong 2-amalgamation of $\hat{\mathbf{K}}^r$ of structures of size bounded by \aleph_s implies that every $M \in \hat{\mathbf{K}}^r$ of size \aleph_s has an extension $N \in \mathbf{At}^r$, which is also of size \aleph_s . [To see this, use the characterization of \mathbf{At}^r given in Corollary 4.2 and construct N as a union of a chain, each time taking one step toward richness.]

Arguing similarly, we see that for $s \leq r - 2$ the class \mathbf{At}^r has disjoint (not strongly disjoint) amalgamation in \aleph_s . From this it follows that \mathbf{At}^r has an element of size \aleph_{s+1} and, moreover, every $N \in \mathbf{At}^r$ of size \aleph_{s+1} has a proper extension. Thus, \mathbf{At}^r has an element of size \aleph_{s+2} . Taking s as $r - 2$ gives the second sentence. $\square_{4.3}$

Remark 4.4 (Other amalgamation notions). Clearly strong disjoint amalgamation implies disjoint amalgamation and the argument of Claim 3.4 works for either. {apprem}

We say M is *strongly embedded* in N if there is an isomorphism f such that $f[M] \prec_{\mathbf{K}} N$. The more usual notion of the 2-amalgamation property is given by replacing in Definition 3.2 ‘ N_u is a substructure of M ’ by ‘each N_u is strongly embedded in M ’. For $n > 2$ there are more complicated conditions for ‘independent amalgamation’, to maintain the conditions on coherence of maps, which are described in [1, 13] and somewhat less technically in [3]. Disjoint amalgamation implies amalgamation but as we will now see, the converse fails.

We use a slight modification of Lemma 2.3 of [4] to obtain our negative results. The proof of the following Fact is included as a convenience for the reader. {combl}

Fact 4.5. *For every $k \in \omega$, if cl is a locally finite closure relation on a set X of size \aleph_k , then there is an independent subset of size $k + 1$.*

Proof. By induction on k . When $k = 0$, take any singleton not included in $\text{cl}(\emptyset)$. Assuming the Fact for k , given any locally finite closure relation cl on a set X of size \aleph_{k+1} , fix a cl -closed subset $Y \subseteq X$ of size \aleph_k and choose any $a \in X \setminus Y$. Define a locally finite closure relation cl_a on Y by $\text{cl}_a(Z) = \text{cl}(Z \cup \{a\}) \cap Y$. It is easily checked that if $B \subseteq Y$ is cl_a -independent, then $B \cup \{a\}$ is cl -independent. $\square_{4.5}$ {allmax}

Lemma 4.6. *For every $r \geq 1$, $\hat{\mathbf{K}}^r$*

1. *has only maximal models in \aleph_r and so (disjoint) 2-amalgamation is trivially true in \aleph_r ;*
2. *fails 2-amalgamation in \aleph_{r-1} .*

Proof. Fix $M \in \hat{\mathbf{K}}^r$ of size \aleph_r and assume by way of contradiction that it had a proper extension $N \in \hat{\mathbf{K}}^r$. Let cl denote the locally finite closure relation on N given by the family of functions $\{f_n\}$. As in the proof of Fact 4.5, fix $a \in N \setminus M$ and define a closure relation cl_a on M by $\text{cl}_a(Z) = \text{cl}(Z \cup \{a\}) \cap M$. As cl_a is locally finite, it follows from Fact 4.5 that there is a cl_a -independent subset $B \subseteq M$ of size $r + 1$. But then, $B \cup \{a\}$ would be a cl -independent subset of N of size $r + 2$, contradicting $N \in \hat{\mathbf{K}}^r$.

For the second part, we just modify this argument and show that an amalgamation problem (M_0, M_1, M_2) of elements from $\hat{\mathbf{K}}^r$ of size \aleph_{r-1} with M_0 a substructure of both M_1 and M_2 is solvable if and only if M_i is embeddable into M_{3-i} over M_0 for either $i = 1$ or 2 . To see the non-trivial direction, suppose a triple (M_0, M_1, M_2) are chosen as above and there is $M_3 \in \hat{\mathbf{K}}^r$ and embeddings $f: M_1 \rightarrow M_3$ and $g: M_2 \rightarrow M_3$, each over M_0 , with elements $f(a_1) \notin g(M_2)$ and $g(a_2) \notin f(M_1)$. We obtain a contradiction by considering the closure relation on M_0 defined by

$$\text{cl}^*(Z) = \text{cl}(Z \cup \{f(a_1), g(a_2)\}) \cap M_0$$

with the latter closure relation cl computed in M_3 . As cl^* is locally finite, it follows from Fact 4.5 that there is a cl^* -independent subset $B \subseteq M_0$ of size r . We obtain a contradiction to $M_3 \in \hat{\mathbf{K}}^r$ by showing that the set $B \cup \{f(a_1), g(a_2)\}$ is cl -independent. To see this, first note that since $B \cup \{f(a_1)\} \subseteq f(M_1)$, $g(a_2) \notin \text{cl}(B \cup \{f(a_1)\})$. Similarly, $f(a_1) \notin \text{cl}(B \cup \{g(a_2)\})$. But, for any $b \in B$, the cl^* -independence of B implies that $b \notin \text{cl}((B \setminus \{b\}) \cup \{f(a_1), g(a_2)\})$, completing the argument characterizing which amalgamation problems are solvable.

To complete part 2) we must establish (*): There exists a triple (N_0, N_1, N_2) consisting of elements of $\hat{\mathbf{K}}^r$, each of size \aleph_{r-2} such that for $i = 1, 2$, each of N_{3-i} contains an element whose type over N_0 is not realized in the other.

For this, choose $M \in \hat{\mathbf{K}}^r$ of cardinality \aleph_{r-1} and two finite structures $M_1, M_2 \in \hat{\mathbf{K}}^r$ such that $M_1 \cap M_2 = M_0$ but M_1 and M_2 are not isomorphic over N_0 . Now disjointly amalgamate each M_i with M over N_0 . The resulting triple is as required. $\square_{4.6}$

Part 1) of Lemma 4.6 extends immediately to structures in \mathbf{At}^r . We now give the considerably more involved argument that \mathbf{At}^r fails 2-amalgamation in \aleph_{r-1} . The difficulty is that we must establish (*) for \mathbf{At}^r rather than $\hat{\mathbf{K}}^r$. For most of this discussion, we work with the class $\hat{\mathbf{K}}^r$, passing to \mathbf{At}^r only at the end. We require the following new notion.

Definition 4.7. Fix any infinite $U \subseteq \omega$. For any $s \leq r - 2$, and any s -system $\langle N_u : u \in \mathcal{P}^-(s) \rangle$ from $\hat{\mathbf{K}}^r$, a U -amalgam is any strong, disjoint amalgam M with the additional property that for every unspecified $(r + 1)$ -tuple \mathbf{b} , if $M \models R_t(\mathbf{b})$, then $t \in U$. (As in the proof of Theorem 4.1, unspecified means here that \mathbf{b} is not contained in any element of the system.)

Lemma 4.8. For any infinite $U \subseteq \omega$, Theorems 4.1 and 4.3 go through word for word, using ‘ U -amalgamation’ in place of ‘strong, disjoint amalgamation’. In particular, by Theorem 4.3, any triple (N_0, N_1, N_2) of elements of $\hat{\mathbf{K}}^r$ with $N_1 \cap N_2 = N_0$, each of size at most \aleph_{r-2} has a U -amalgamation. {U}

Definition 4.9. Suppose $N \in \hat{\mathbf{K}}^r$, $B \subseteq N$, and $a \in N \setminus B$. A relevant $(r + 1)$ -tuple is an element of $(B \cup \{a\})^{r+1}$ with exactly one occurrence of a . We define the species of a over B in N .

$$\text{sp}_N(a/B) = \{t \in \omega : N \models R_t(\mathbf{c}) \text{ for some relevant } \mathbf{c}\}$$

The subscript N is necessary since while the species of a/B is a property of sequences from $B \cup \{a\}$, the predicates are specified on those elements only by the model N . The next few lemmas describe basic constructions with U -amalgamation. The definition of U -amalgamation allows us to ‘extend the base’.

Lemma 4.10. Fix any infinite subset $U \subseteq \omega$. For any $N_0 \subseteq N_1$ and $N_0 \subseteq N'_0$, all from $\hat{\mathbf{K}}^r$, with $N_1 \cap N'_0 = N_0$, every U -amalgam $N'_1 \in \hat{\mathbf{K}}^r$ satisfies for every $a \in N_1 \setminus N_0$:

$$\text{sp}_{N'_1}(a/N'_0) \setminus \text{sp}_{N_1}(a/N_0) \subseteq U.$$

Definition 4.11. Let $M \subseteq M_1 \subseteq M_2$ each be in $\hat{\mathbf{K}}^r$ and $V \subseteq \omega$. We write $M_1 \prec_{M,V} M_2$ if for each $a \in M_2 - M_1$, $\text{sp}_{M_2}(a/M) \cap V$ is finite.

Lemma 4.12. Fix infinite disjoint $U, V \subseteq \omega$. Suppose $N_1 \in \hat{\mathbf{K}}^r$ is of size at most \aleph_{r-2} , $M \subseteq M' \in \mathbf{K}^r$ (hence finite) with $M' \cap N_1 = M$. Then any U -amalgam N_2 of M' and N_1 over M satisfies: $N_1 \prec_{M,V} N_2$.

Proof. If $a \in N_1$, there are no new relevant tuples to be assigned values. Any $a \in N_2 \setminus N_1$ is in $M' \setminus M$. The only sequences that are not assigned a value from U are from $M \cup \{a\} \subseteq M'$, and there are only finitely many of them. $\square_{4.12}$

Lemma 4.13. Fix an infinite-coinfinite $U \subset \omega$. Suppose $M_0 \subseteq M_1$ are in $\hat{\mathbf{K}}^r$ and have cardinality at most \aleph_{r-2} . Then there is an $M^* \in \mathbf{At}^r$ such that for any $V \subset \omega$ that is disjoint from U , $M_1 \prec_{M_0,V} M^*$

Proof. We obtain M^* by modifying the induction of the second paragraph of the proof of Theorem 4.3 to build a sequence of models M_α with union M^* . At each stage α , U -amalgamate a finite $A \in \mathbf{K}^r$ with M_α over $A \cap M_\alpha$. Then for any $a \in M^*$, it first appears in some M_α and the species of a/M_0 in M_β is defined only for $\beta \geq \alpha$. Further, for any $\beta > \alpha$, $\text{sp}_{M^*}(a/M_0) = \text{sp}_{M_\beta}(a/M_0) = \text{sp}_{M_\alpha}(a/M_0)$ and $\text{sp}_{M_\alpha}(a/M) \cap V$ is finite. It is easy to organize the construction so $M^* \in \mathbf{At}^r$. $\square_{4.13}$

We now define the notion of obstruction (to amalgamation). Of course, we can overcome that obstruction in cardinals at most \aleph_{r-2} . But we will construct an obstruction of cardinality \aleph_{r-1} which by Lemma 4.6 cannot be overcome.

Definition 4.14. Fix pairwise disjoint, infinite $U, V, W \subseteq \omega$. An *obstruction* is a triple (N_0, N_1, N_2) consisting of elements of $\hat{\mathbf{K}}^r$, such that $N_1 \cap N_2 = N_0$ and

- There is exactly one $a \in N_1 \setminus N_0$ such that $\text{sp}_{N_1}(a/N_0)$ has infinite intersection with V and $N_0 \prec_{N_0,V} N_2$. (That is, for any $b \in N_2 \setminus N_0$, $\text{sp}_{N_1}(b/N_0) \cap V$ is finite.)
- There is exactly one $b \in N_2 \setminus N_1$ that $\text{sp}_{N_2}(b/N_0)$ has infinite intersection with W and $N_0 \prec_{N_0,W} N_1$. (That is, and for any $a \in N_1 \setminus N_2$, $\text{sp}_{N_1}(a/N_0) \cap W$ is finite.)

An obstruction is said to have cardinality κ if each N_i has cardinality κ . An *extension* of an obstruction (N_0, N_1, N_2) is an obstruction (N'_0, N'_1, N'_2) satisfying $N_i \subseteq N'_i$ and $N'_i \cap (N_1 \cup N_2) = N_i$ for each i . An extension is *proper* if all three of the structures increase. An *atomic obstruction* is an obstruction (M_0, M_1, M_2) in which every M_i is in \mathbf{At}^r .

‘Obstruction’ has a hidden parameter: ‘with respect to disjoint V, W ’. In order to work with the notion we need a third set U disjoint from each of them. **We will keep fixed disjoint, infinite subsets U, V, W of ω in the following construction.** Now we will create obstructions.

Lemma 4.15. For any $N_0 \in \hat{\mathbf{K}}^r$ of size \aleph_0 , there is a 1-point extension $N_1 \in \hat{\mathbf{K}}^r$ with universe $N_0 \cup \{a\}$ such that $\text{sp}_{N_1}(a/N_0)$ is an infinite subset of V .

Proof. Fix any nested sequence $\langle M_k : k \in \omega \rangle$ of elements of \mathbf{K}^r (hence finite) such that $N_0 = \bigcup_k M_k$. Inductively assume that we have constructed an element $M_{k,a}$ of \mathbf{K} with universe $M_k \cup \{a\}$ satisfying $\text{sp}_{M_{k,a}}(a/M_k) \subseteq V$ and $|\text{sp}_{M_{k,a}}(a/M_k)| \geq k$. Fix any element $t^* \in V$ that is larger than both $\max\{\text{sp}_{M_{k,a}}(a/M_k)\}$ and $\|M_{k+1}\|$. Then construct an extension with universe $M_{k+1} \cup \{a\}$ using the procedure in the proof of Theorem 4.1, saying that $R_{t^*}(\mathbf{b})$ holds for every unspecified $(r+1)$ -tuple. $\square_{4.15}$

{start}

Lemma 4.16. *For every $N \in \hat{\mathbf{K}}^r$ of size \aleph_0 , there is a 2-point extension N^* with universe $N \cup \{a, b\}$ such that*

- $\text{sp}_{N^*}(a/N)$ is an infinite subset of V ; and
- $\text{sp}_{N^*}(b/N)$ is an infinite subset of W .

Thus there is a countable obstruction in $\hat{\mathbf{K}}^r$.

Proof. Apply Lemma 4.15 with V and W to create two one point extensions and then use U -amalgamation to construct N^* . The required obstruction is (N, Na, Nb) . $\square_{4.16}$

{chain}

Lemma 4.17. *Suppose $r \geq 2$. Any obstruction (N_0, N_1, N_2) of cardinality $\leq \aleph_{r-2}$ has a proper atomic extension.*

Proof. Let (N_0, N_1, N_2) be an obstruction with U -amalgam N^* . Choose by Theorem 4.3 a proper extension $N_4 \in \mathbf{At}^r$ of N_0 such that $N_4 \cap N^* = N_0$. Let N_7 be a U -amalgam of N_4 and N^* ; it contains U -amalgams N_5 of N_4 and N_1 and N_6 of N_4 and N_2 . By Lemma 4.10, (N_4, N_5, N_6) is an obstruction. Apply Lemma 4.13 twice to construct $N'_5, N'_6 \in \mathbf{At}^r$ such that $N_5 \prec_{N_4, V} N'_5$ and $N_6 \prec_{N_4, W} N'_6$. Then, (N_4, N'_5, N'_6) is as required. $\square_{4.17}$

{atob}

Lemma 4.18. *For $r \geq 1$ there is an atomic obstruction (M_0, M_1, M_2) of cardinality \aleph_{r-1} .*

Proof. Build by Lemma 4.17 an increasing \aleph_{r-1} chain of atomic obstructions. To get started, use Lemma 4.16. $\square_{4.18}$

{atfail}

Proposition 4.19. *For any $r \geq 1$, \mathbf{At}^r fails amalgamation in \aleph_{r-1} .*

Proof. By Lemma 4.18 when $r \geq 2$, we have an atomic obstruction (M_0, M_1, M_2) of cardinality \aleph_{r-1} . For such an obstruction, M_i does not embed into M_{3-i} over M_0 for $i = 1, 2$, so by the argument justifying the second sentence of 4.6, this triple cannot be amalgamated into any element of $\hat{\mathbf{K}}^r$, much less an element of \mathbf{At}^r . $\square_{4.19}$

Now we sum up the properties of the example.

{sumup}

Theorem 4.20. *For every $r \geq 1$, the class \mathbf{At}^r satisfies:*

1. *there is a model of size \aleph_r , but no larger models;*
2. *every model of size \aleph_r is maximal, and so (disjoint) 2-amalgamation is trivially true in \aleph_r ;*
3. *disjoint 2-amalgamation holds up to \aleph_{r-2} ;*
4. *2-ap fails in \aleph_{r-1} .*

Proof. By Theorem 4.3, Lemma 4.6, and Proposition 4.19. $\square_{4.20}$

This improves with a much simpler argument the result of [5] that found examples of abstract elementary classes with disjoint amalgamation up to \aleph_{k-3} but no model of cardinality $> \beth_k$.

Recall from e.g. [14] the definition of homogenous characterizability.

{newdefhom}

Definition 4.21. κ is homogeneously characterized by a complete sentence ϕ_κ of $L_{\omega_1, \omega}$ if ϕ_κ characterizes κ and there is a predicate V such that V defines a set of absolute indiscernibles in the countable model (every permutation of $V(M)$ extends to an automorphism of M) and there is a model N with $|V(N)| = \kappa$.

In order to find the homogeneous characterization we modify the class \mathbf{K}^r slightly. We extend the vocabulary with unary predicates U and V , and a unary function p . The additional axioms assert that p projects U onto V ; the $(r+1)$ -ary functions f_n and relations R_n are defined only on U and satisfy the axioms of Section 2 on U . We let \mathbf{K}_h^r denote the class of finite structures in the expanded language satisfying these constraints. By e.g., [4], the class \mathbf{K}_h^r satisfies disjoint amalgamation and thus has a Fraïssé limit M_h . Let ϕ_h^r denote the Scott sentence of M_h .

{homchar}

Theorem 4.22. The sentence ϕ_h^r characterizes \aleph_r homogeneously.

Proof. Use disjoint amalgamation as in the proof of Theorem 1.10 of [4] to show that $V(M_h)$ is a set of absolute indiscernibles, and then apply Claim 3.4, noting that the disjoint amalgamation guarantees that the interpretation of the predicate V has full cardinality in every model. $\square_{4.22}$

Remark 4.23. The number of models in the characterized cardinal We noted above that if an AEC has disjoint amalgamation in \aleph_s it has a model in \aleph_{s+2} . Thus on general grounds we knew $\hat{\mathbf{K}}^r$ fails disjoint amalgamation in \aleph_{r-1} . But to show amalgamation failed we had to use our particular combinatorics in Lemma 4.6.2. We don't have a 'soft' argument that 'ordinary' amalgamation must fail in \aleph_{r-1} . But, if an AEC has the amalgamation property in κ and all models in κ^+ are maximal, the models in κ^+ can be amalgamated on a model of size κ .⁴ Then there is a 1-1 map from models of cardinality κ^+ to models of cardinality κ : Map M of cardinality κ^+ to a submodel M' of cardinality κ . If M and N map to the same model, they have a common extension. But both are maximal, so they must be isomorphic. So the number of models in \aleph_r is no more than the number in \aleph_{r-1} .

We show that the anomalous situation of the last remark does not hold in our examples. Since no really new techniques are introduced and there could be many variants on the methods here, we just sketch the argument.

Remark 4.24. We show each of the classes $\hat{\mathbf{K}}^r$ and \mathbf{At}^r have 2^{\aleph_s} models in \aleph_s for $1 \leq s \leq r$. In addition, $\hat{\mathbf{K}}^r$ has 2^{\aleph_0} models in \aleph_0 . As in the earlier arguments in this paper one first proves the result for $\hat{\mathbf{K}}^r$. Relativizing the construction to permit indices on the R_n only for n in a fixed infinite $W \subset \omega$ gives a situation that differs from the original only by notation. So any family of distinct sets W_i for $i < \omega$ gives continuum many countable models of $\hat{\mathbf{K}}^r$. Ad hoc arguments or the existence of algebraic chains in the sense of (Baldwin, Laskowski, Shelah forthcoming) shows there are 2^{\aleph_1} models of $\hat{\mathbf{K}}^r$ in \aleph_1 . Now induction on s using properties of U -amalgamation shows $\hat{\mathbf{K}}^r$ has 2^{\aleph_s} models in \aleph_s for $1 \leq s \leq r$. Finally, when $s > 1$ a similar argument allows one to close non-isomorphic $M, N \in \hat{\mathbf{K}}^r$ to non-isomorphic models in \mathbf{At}^r . The last two arguments rely on the notion of a V -component: A V -component E of $a \in N$ is a maximal uncountable subset of N such that for any tuple $\mathbf{e} \in E$ if every permutation \mathbf{f} of $\mathbf{e}a$ satisfies $R_t(\mathbf{f})$ then $t \in E$.

⁴The straightforward argument is written out for \aleph_1 in [6].

Question 4.25. *Is there a (complete) sentence of $L_{\omega_1, \omega}$ which characterizes κ and has fewer than 2^κ models of cardinality κ ?*

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