A COMPLICATED ω-STABLE DEPTH 2 THEORY

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Abstract. We present a countable complete first order theory T which is model theoretically very well behaved: it eliminates quantifiers, is ω-stable, it has NDOP and is shallow of depth two. On the other hand, there is no countable bound on the Scott heights of its countable models, which implies that the isomorphism relation for countable models is not Borel.

Keywords: omega-stability, classification, countable models, Borel reducibility, Scott height

§1. Introduction. This paper presents the main result of the thesis [10] directed by E. Bouscaren. It is a continuation of the article [9]. For the reader's convenience we will recall here the key definitions. For definitions and results in Shelah's classification theory, see [13] and [1].

- A non-algebraic type $p \in S(A)$ is *strongly regular* if there is a formula (with parameters) such that any type $q \in S(B \cup A)$ containing that formula is either orthogonal to p or a non-forking extension of p.
- A type p is non-orthogonal to a set C (notation $p \not\perp C$), if there is some $q \in S(C)$ with $p \not\perp q$.
- An ω-stable theory has the NDOP if for all four models M_i ($i < 4$) such that $M_0 \subset M_1 \cap M_2$, M_1 independent from M_2 over M_0 and M_3 prime over $M_1 \cup M_2$, and any strongly regular $p \not\perp M_3$, either $p \not\perp M_1$ or $p \not\perp M_2$.
- A type $p \in S(A)$ (A finite) is ENI if it is strongly regular and there is some finite $B \supset A$ such that a non-forking extension of p to B is non-isolated. $p \in S(C)$ for infinite C is ENI if there is a finite A such that p does not fork over A and $p \restriction A$ is ENI. p is NENI is it is strongly regular and not ENI.
- We define the ENI-NDOP similarly to the NDOP but demanding the described property only for ENI-types over M_3 instead of all strongly regular types. Thus, the NDOP implies the ENI-NDOP.

It is known (see [13] or [1]) that the NDOP allows tree decompositions of all models. One result described in [9] is that the ENI-NDOP is enough to have tree decompositions for all *countable* models. Along with those tree decompositions come notions of depth (for types and for the theory itself) defined in terms of

2000 Mathematics Subject Classification: 03C15, 03C45, 03E15

foundation ranks for those decomposition trees. We recall our definitions of eni-depth and ENI-depth:

Definition 1.1. *A stationary type* $p \in S(A)$ *is said to* support *another type* q*, if there is a model* $M \supset A$ *and* $a \models p|M$ *such that* $q \perp M$ *and* $q \not\perp M[a]$ *(where* $M[a]$ *is a model prime over* $M \cup \{a\}$*).*

Definition 1.2.
- $\mathrm{ENI}-\mathrm{dp}(p) \geq 0$ *for all* p
- *For limit* α*,* $\mathrm{ENI}-\mathrm{dp}(p) \geq \alpha$ *if* $\forall \beta < \alpha$ $\mathrm{ENI}-\mathrm{dp}(p) \geq \beta$
- $\mathrm{ENI}-\mathrm{dp}(p) \geq \alpha + 1$ *if* p *is ENI and supports an ENI type* q *with* $\mathrm{ENI}-\mathrm{dp}(q) \geq \alpha$

Then we set $\mathrm{ENI}-\mathrm{dp}(p) = \infty$ *if* $\mathrm{ENI}-\mathrm{dp}(p) \geq \alpha$ *for all* α*, otherwise* $\mathrm{ENI}-\mathrm{dp}(p) = \min\{\alpha | \mathrm{ENI}-\mathrm{dp}(p) \geq \alpha$ *and* $\mathrm{ENI}-\mathrm{dp}(p) \not\geq \alpha + 1\}$

Definition 1.3.
- $\mathrm{eni}-\mathrm{dp}(p) \geq 0$ *for all* p
- $\mathrm{eni}-\mathrm{dp}(p) \geq 1$ *if* p *supports an ENI type*
- *For limit* α*,* $\mathrm{eni}-\mathrm{dp}(p) \geq \alpha$ *if* $\forall \beta < \alpha$ $\mathrm{eni}-\mathrm{dp}(p) \geq \beta$
- $\mathrm{eni}-\mathrm{dp}(p) \geq \alpha + 1$ *if* p *supports a* q *with* $\mathrm{eni}-\mathrm{dp}(q) \geq \alpha$

Again we set $\mathrm{eni}-\mathrm{dp}(p) = \infty$ *if* $\mathrm{eni}-\mathrm{dp}(p) \geq \alpha$ *for all* α*, otherwise* $\mathrm{eni}-\mathrm{dp}(p) = \min\{\alpha | \mathrm{eni}-\mathrm{dp}(p) \geq \alpha$ *and* $\mathrm{eni}-\mathrm{dp}(p) \not\geq \alpha + 1\}$

Definition 1.4. *Let* A *be the set of all types realized in tree decompositions of models of* T*. Then*
- $\mathrm{ENI}-\mathrm{dp}(T) = \sup\{\mathrm{ENI}-\mathrm{dp}(p) + 1 | p \in A\}$
- $\mathrm{eni}-\mathrm{dp}(T) = \sup\{\mathrm{eni}-\mathrm{dp}(p) + 1 | p \in A\}$

We have seen in [9] that the ENI-depth of T appears to be unrelated to the complexity of the class of countable models of T, whereas we could show a few results involving the notion of eni-depth. We proved that theories with eni-depth 1 have an uncomplicated classification problem (concerning their countable models), their isomorphism relation for countable models was *smooth*, i.e. Borel reduces to the equality relation on the real numbers. This refers to the following basic definition:

Definition 1.5. *For* L *countable and* T *an* L*-theory (or more generally an* $L_{\omega_1\omega}$*-theory), the class of all countable models of* T *has the structure of a standard Borel space* X_T *("standard" means that the Borel structure is induced by a Polish topology, i.e. one that is separable and completely metrizable).*

Let $\cong_T$ *denote the isomorphism relation on* X_T*. Then for* T_1 *and* T_2 *two theories, we say* $\cong_{T_1}$ *Borel reduces to* $\cong_{T_2}$ *(notation* $\cong_{T_1} \leq_B \cong_{T_2}$*) if there is a Borel map* $f : X_{T_1} \to X_{T_2}$ *such that for* $a, b \in X_{T_1}$*,* $a \cong b$ *if and only if* $f(a) \cong f(b)$*.*

The relation $\leq_B$ is a partial preordering which has been investigated (in even greater generality: not only for isomorphism relations but more general equivalence relations) in descriptive set theory (see for example [3], [5], [6], [4], [7], [8], [12]). $\cong_{T_1} \leq_B \cong_{T_1}$ means that the classification problem for countable models of T_1 is at most as complicated as that for T_2. $\cong_T$ is called Borel if it is Borel as a subset of $X_T \times X_T$. It can be shown that $\cong_T$ is always analytic, but non-Borel examples exist. Smooth theories have a Borel isomorphism.

Our main question is how the notion of eni-depth is related to $\leq_B$. Since eni-depth 1 theories are very low in the ordering $\leq_B$, a natural next step is to ask how complicated eni-depth 2 theories can be. In the following we will describe an example T of eni-depth 2 with a non-Borel isomorphism. Besides the fact that equality on countable sets of reals can be reduced to $\cong_T$, we do not know how high its isomorphism relation is in the ordering $\leq_B$. In particular we do not know if it is *Borel-complete*, i.e. if all isomorphism relations can be reduced to it (as is the case for the theory of arbitrary graphs). No first order axiomatizable theory with non-Borel and not Borel-complete isomorphism relation is known so far.

To prove the non-Borelness of $\cong_T$, we will use a result of [2] which states that this is equivalent to the non-existence of a countable bound for the Scott heights of all countable models of T.

As is shown in [11], *classifiable* theories (in the sense of Shelah) have an ordinal bound on the Scott heights (for all models, not only countable), so this is true for our theory T. However, T is a counterexample to a conjecture stated in the same article that this bound should be countable.

§2. Definition of the theory and basic properties.

Let $L = \{U, C_i, V_i, \pi_i^j, S_i\}_{i<\omega, j\leq i+1}$ be a multi-sorted language with infinitely many sorts U, V_i, C_i $(i < \omega)$, where $\pi_i^j : V_i \to C_j$ (for all $j \leq i$), $\pi_i^{i+1} : V_i \to U$ and $S_i : V_i \to V_i$ are unary function symbols.

As a notation, let π_i be the function which to $x \in V_i$ assigns the $(i+2)$-tuple $(\pi_i^0(x), \ldots, \pi_i^{i+1}(x))$ (the symbols π_i are not part of our language L). Let T be the L-theory axiomatized as follows :

(1) U contains infinitely many elements and each C_i contains exactly two elements.
(2) All S_i $(i < \omega)$ define successor functions in V_i (i.e. bijective functions without cycles)
(3) π_i is a function from V_i *onto* $C_0 \times C_1 \times \cdots \times C_i \times U$ for all $i < \omega$.
(4) For all $i < \omega$, $\pi_i \circ S_i = \pi_i$, i.e. the π_i-fibers are the union of connected S_i components (that are isomorphic to $\mathbb{Z}$ with successor function)

It is straightforward to see that T is complete and eliminates quantifiers.

We will now give a description of the 1-types of the theory. There is (up to non-forking extensions) only one 1-type saying that x belongs to U. All types saying that x belongs to some C_i are algebraic. Actually, the algebraic closure of the empty set is exactly $\bigcup_{i<\omega} C_i$. **We enumerate the elements of the sets C_i: let** $C_i = \{a_i^0, a_i^1\}$ (which does *not* mean that we add them as constants to our language). Now, given $b \in U$, there are exactly 2^{i+1} different complete 1-types over $\{b\} \cup \operatorname{acl}(\emptyset)$ containing the formula $V_i(x)$, namely for all sequences $s \in 2^{i+1}$ the type p_s^b that says $\pi_i(x) = (a_0^{s(0)}, a_1^{s(1)}, \ldots, a_i^{s(i)}, b)$. Those types are mutually conjugate, stationary, strongly regular and multidimensional and their dimension in a model equals the number of connected S_i-components of elements of that type. The p_s^b are supported by the non-multidimensional strongly regular type $t(b/\emptyset)$.

From our understanding of the 1-types follows:

PROPOSITION 2.1. *T is ω-stable, has NDOP and is shallow of depth two.*

REMARK 2.2. *T has also the ENI-NDOP (since NDOP implies ENI-NDOP) and we can verify that T has eni-depth two and ENI-depth one. Actually, the type saying "x is in U" is NENI and supports the ENI types p_s^b described above which, themselves, do not support any other type.*

DEFINITION 2.3. *Let $2^{<\omega}$ denote the set of finite sequences of 0's and 1's and let $\mathcal{A}$ be the set of functions $f : 2^{<\omega} \setminus \{\emptyset\} \to \omega+1$ ("countably labelled full binary trees"). If $M \models T$ is countable and $b \in U(M)$, we define $\delta_b^M \in \mathcal{A}$ by*

$$\delta_b^M(s) = \begin{cases} \dim_M(p_s^b) - 1 & \text{if } \dim_M(p_s^b) < \omega \\ \omega & \text{if } \dim_M(p_s^b) = \omega \end{cases}$$

(note that we must always have $\dim_M(p_s^b) \geq 1$).

DEFINITION 2.4. *Let $\mathcal{C}$ denote the* Cantor group*, i.e. topological Cantor space 2^ω equipped with componentwise addition modulo 2. This defines a polish group. $\mathcal{C}$ acts on $\mathcal{A}$ in the following way:*

$$\text{For } \sigma \in \mathcal{C} \text{ and } \delta \in \mathcal{A} \text{ let} \quad \sigma\delta(s) = \delta(s + \sigma \restriction |s|)$$

If $\sigma(n) = 1$ for some $\sigma \in \mathcal{C}$ and $n < \omega$, this action "flips" completely the nth level of the labeled tree $\delta \in \mathcal{A}$. Thus all elements of one orbit are isomorphic labeled trees, but there are isomorphic trees in distinct orbits. It follows that there are $2^{\aleph_0}$ orbits and orbits can have cardinality 2^κ for all $\kappa \leq \aleph_0$.

The following theorem characterizes isomorphism for countable models of T (it also implies that there are $2^{\aleph_0}$ isomorphism types of countable models).

THEOREM 2.5. *Countable models $M, N \models T$ are isomorphic and only if there exists $\sigma \in \mathcal{C}$ and a one-one function f from $U(M)$ onto $U(N)$ such that for all $b \in U(M)$, $\delta_b^M = \sigma\delta_{f(b)}^N$.*

§3. Non-Borelness. The rest of this paper is devoted to an exposition of our proof that the Scott ranks of countable models of T have no countable bound. We omit some of the easier and straightforward proofs of lemmas and propositions. Full details can be found in [10]. We will have to introduce a rather large amount of notation; to help the reader we included a list of symbols which can be found in the end of the paper. We start by recalling the definition of Scott height:

DEFINITION 3.1. *Let T' be a theory, $M, N \models T'$ and $\bar{a} \in M^n$, $\bar{b} \in N^n$ for some $n < \omega$. We define recursively the equivalence $(M, \bar{a}) \equiv_\alpha (N, \bar{b})$ ($\alpha < \omega_1$):*

- *$(M, \bar{a}) \equiv_0 (N, \bar{b})$ if and only if $\bar{a}$ and $\bar{b}$ realize the same type without quantifiers over $\emptyset$.*
- *if α is a limit ordinal, we have $(M, \bar{a}) \equiv_\alpha (N, \bar{b})$ if and only if for all $\beta < \alpha$ $(M, \bar{a}) \equiv_\beta (N, \bar{b})$.*

- $(M,\bar{a}) \equiv_{\alpha+1} (N,\bar{b})$ *if and only if* $\forall a \in M \exists b \in N (M,\bar{a}a) \equiv_\alpha (N,\bar{b}b)$ *and* $\forall b \in N \exists a \in M (M,\bar{a}a) \equiv_\alpha (N,\bar{b}b)$

Then we define $M \equiv_\alpha N$ *by* $(M,\emptyset) \equiv_\alpha (N,\emptyset)$,

$$\mathrm{SH}(M) = \min\{\alpha < \omega_1 | \forall \bar{a},\bar{b}[(M,\bar{a}) \equiv_\alpha (M,\bar{b}) \to (M,\bar{a}) \equiv_{\alpha+1} (M,\bar{b})]\}$$

(the Scott height *of* M*) and*

$$\mathrm{SH}(T') = \sup\{\mathrm{SH}(M)|M \models T'\}$$

(which can be ω_1 *or a countable ordinal).*

In order to prove the theorem

Theorem 3.2. $\mathrm{SH}(T) = \omega_1$

we proceed in five steps :

(I) Restricting our attention to a particular subclass of models of our theory.
(II) Defining a nice class of pairs of "extended" models (models with some distinguished finite tuple of elements).
(III) Defining objects we call *configurations* (basically ω-sequences of subsets of the Cantor space) that code the "difference" between two such extended models.
(IV) Translating properties of pairs of extended models to properties of configurations. *Thin* configurations will correspond to non-isomorphic models and α-*rich* configurations to $(\alpha+\omega)$-equivalent models.
(V) Constructing inductively for every $\alpha < \omega_1$ configurations which are both α-rich and thin. This will ensure the existence of countable models $M_\alpha, N_\alpha \models T$ such that $M_\alpha \not\cong N_\alpha$ and $M_\alpha \equiv_\alpha N_\alpha$. Consequently, $\mathrm{SH}(T) > \alpha$ for all $\alpha < \omega_1$.

Details for (I)-(V):

(I) For $\delta \in \mathcal{A}$ let $\mathrm{Stab}(\delta) = \{\sigma \in \mathcal{C} | \sigma\delta = \delta\}$ its *stabilizer* which is a closed subgroup of $\mathcal{C}$. Since $\mathcal{C}$ is abelian, $\mathrm{Stab}(\delta)$ actually only depends on the *orbit* $\mathcal{C}\delta = \{\sigma\delta | \sigma \in \mathcal{C}\}$ of δ. The notion of stabilizer will be used in the definition of (A1) below.

If M is a model of T and $\delta \in \mathcal{A}$ we define the *multiplicity* of δ in M as the number of times that tree is realized in M:

$$\mu(\delta, M) = |\{b \in U(M) | \delta_b^M = \delta\}|$$

Definition 3.3. *Let* $\delta \in \mathcal{A}$ *and* $o = \mathcal{C}\delta$ *be the corresponding orbit. For any model* M *Let* M^o *be the union of* $\bigcup_{i<\omega} C_i$, $\{b \in U(M) | \delta_b^M \in o\}$ *and* $\bigcup_{i<\omega} \{d \in V_i(M) | \delta^M_{\pi_i^{i+1}(d)} \in o\}$.

If for some model M, and some orbit $o = \mathcal{C}\delta$, the set $\{b \in U(M) | \delta_b^M \in o\}$ is infinite, M^o is the universe of a submodel of M (with the induced structure) which we denote by M^o as well.

Definition 3.4. *Given an orbit $o = \mathcal{C}\delta$, we set*

$$S^{M,o} = \{\sigma \in \mathcal{C} | \forall b \in U(M)(\delta_b^M \in o \rightarrow \mu(\sigma\delta_b^M, M) = \mu(\delta_b^M)\})$$

This is the set of permutations $\sigma \in \mathcal{C}$ that preserve multiplicities for realizations of o and defines a subgroup of $\mathcal{C}$ (which clearly contains $\mathrm{Stab}(\delta)$).

We have the following characterization of $S^{M,o}$: for all σ, $\sigma \in S^{M,o}$ if and only if σ witnesses an automorphism of M^o according to Theorem 2.5. I.e. there is an automorphism of M^o which for each $i < \omega$ fixes C_i if and only if $\sigma(i) = 0$. The object $S^{M,o}$ will be used in the definition of (B3) below.

Now we fix an enumeration $(y_i)_{i<\omega}$ of the finite (non-empty) subsets of positive integers $[\omega]^{<\omega} \setminus \{\emptyset\}$ and define a collection $(X_i)_{i<\omega}$ of subsets of ω by

$$n \in X_i \text{ if and only if } i \in y_n$$

This is an *independent family* , i.e. any intersection of finitely many sets of the form X_i and $\omega \setminus X_j$ is non-empty (and therefore actually infinite).

Now, for each $i < \omega$ we can find a $\delta_i \in \mathcal{A}$ with the two following technical properties:

(A1) $\mathrm{Stab}(\delta_i) = \mathcal{C}^{X_i}$, where $\mathcal{C}^{X_i} = \{\sigma \in \mathcal{C} | \forall n < \omega(n \notin X_i \rightarrow \sigma(n) = 0)\}$

(A2) For all n with $0 < n < \omega$ and for all $s \in 2^n$, $\delta_i(s) \geq n$ (lower bound for labels depending on the tree level)

Definition 3.5. *$M \models T$ is* normal *if*

(B1) *M is countable*

(B2) *M realizes exactly the orbits $\mathcal{C}\delta_i$ which we will denote by o_i.*

(B3) *for each $i < \omega$, $\mathcal{C}_0 \subset S^{M,o_i}$ where $\mathcal{C}_0 = \{\sigma \in \mathcal{C} | \exists k < \omega \forall n > k\ \sigma(n) = 0\}$ is the subgroup of sequences that are eventually zero.*

It can be shown that normal models exist and from (A2), (B2) and (B3) follows

Proposition 3.6. *If M, N are normal models and $b_1, b_2, \ldots, b_n \in U(M)$, $b'_1, b'_2, \ldots, b'_n \in U(N)$ pairwise distinct elements then*

$$(M, b_1, b_2, \ldots, b_n) \equiv_\omega (N, b'_1, b'_2, \ldots, b'_n)$$

if and only if there exists some $\sigma \in \mathcal{C}$ such that for all $i \in \{1, \ldots, n\}$, $\delta_{b_i}^M = \sigma\delta_{b'_i}^N$.

(II) For two normal models M, N and an orbit $o = \mathcal{C}\delta$ we let

$$\Delta^{(M,N),o} = \{\sigma \in \mathcal{C} | \forall \delta \in o(\mu(\delta, M) = \mu(\sigma\delta, N))\}$$

If this set is non-empty, it is a coset of $S^{M,o}$ (and $S^{N,o}$ equals $S^{M,o}$ in this case) and it represents exactly the permutations $\sigma \in \mathcal{C}$ witnessing $M^o \cong N^o$ according to Theorem 2.5.

Definition 3.7. – *By an n-*extended model *we understand a pair $(M, \bar{u})$ where $M \models T$ is normal and $\bar{u} \in M^n$.*

– *Given two n-extended models $(M, \bar{u})$, $(N, \bar{v})$ such that $\bar{u}$ and $\bar{v}$ have the same type over the empty set, we define $K^{(M,\bar{u}),(N,\bar{v})} \subset \mathcal{C}$ as the set of* *all elements of* $\sigma \in \bigcup_{i=1}^{n} \Delta^{(M,N),o[u_i]}$ *(where the* u_i *are the components of* $\bar{u}$*) satisfying that for all* $i, j < \omega$*, if* $u_i \in C_j$ *(and therefore also* $v_i \in C_j$*) then* $u_i = v_i$ *if and only if* $\sigma(j) = 0$*.*

To better understand what the set $K^{(M,\bar{u}),(N,\bar{v})}$ represents, we generalize our notion of a submodel M^o realizing only one specific orbit to a submodel $M^{\bar{u}}$ (with $\bar{u}$ a finite tuple in M) consisting of the parts of M that realize orbits "touched" by elements of $\bar{u}$:

Definition 3.8. *Let* $(M, \bar{u})$ *be an* n*-extended model with* $\bar{u} = (u_1, u_2, \ldots, u_n)$*.*

– $o[u_i] = \begin{cases} \emptyset & \text{if } u_i \in C_j \text{ for some } j < \omega \\ \mathcal{C}\delta^M_{u_i} & \text{if } u_i \in U \\ \mathcal{C}\delta^M_{\pi_j^{j+1}(u_i)} & \text{if } u_i \in V_j \text{ for some } j < \omega \end{cases}$

– *Let* $M^{\bar{u}} = \bigcup_{i=1}^{n} M^{o[u_i]}$ *(with the convention* $M^{\emptyset} = \emptyset$*).*

Now it is easy to see that for two n-extended models $(M, \bar{u})$, $(N, \bar{v})$, $K^{(M,\bar{u}),(N,\bar{v})}$ is exactly the set of σ witnessing an isomorphism according to Theorem 2.5 between $M^{\bar{u}}$ and $N^{\bar{v}}$ which maps $\bar{u}$ to $\bar{v}$.

Definition 3.9. *A pair* $((M, \bar{u}), (N, \bar{v}))$ *of* n*-extended models is* regular *if*

(C1) $\bar{u}, \bar{v}$ *have the same type over* $\emptyset$

(C2) *For all* $i < \omega$*,* $M^{o_i} \cong N^{o_i}$ *(i.e.* $\Delta^{(M,N),o_i} \neq \emptyset$*)*

(C3) $K^{(M,\bar{u}),(N,\bar{v})} \neq \emptyset$

(III) We consider regular pairs of n-extended models as the nth step of a back-and-forth. First of all we define the *abstract form* in which we will represent the set of permutations $\sigma \in \mathcal{C}$ still compatible with the back and forth. In the following definition, let $\mathcal{P}(A)$ denote the set of subsets of a set A.

Definition 3.10. *A* configuration *is a triple* (D, X, d) *such that*

– $D : \omega \to \mathcal{P}(\mathcal{C})$*,* $X : \omega \to \mathcal{P}(\omega)$*,* $d \in \mathcal{P}(\mathcal{C})$

– *for each* $n < \omega$*,* $D(n)$ *is empty or a set of the form* $\xi + G + \mathcal{C}^{X(n)}$ *for some* $\xi \in \mathcal{C}$ *and a countable subgroup* $G \subset \mathcal{C}$

– d *is empty or of the form* $\xi + G + \mathcal{C}^{A}$ *for some* $\xi \in \mathcal{C}$*,* $A \subset \omega$ *and a countable subgroup* $G \subset \mathcal{C}$*.*

To a given regular pair $((M, \bar{u}), (N, \bar{v}))$ of n-extended models we assign a configuration $c^{(M,\bar{u}),(N,\bar{v})} = (D, X, d)$ in the following way :

– $D(n) = \Delta^{(M,N),o_n}$

– $X(n) = X_n$

– $d = K^{(M,\bar{u}),(N,\bar{v})}$

Using (B3) and the fact that $\Delta^{(M,N),o_n}$ is a coset of S^{M,o_n} for all $n < \omega$ (because of (C2)) we see that $c^{(M,\bar{u}),(N,\bar{v})}$ is *almost regular* in the sense of the following definition:

Definition 3.11. *A configuration* (D, X, d) *is* almost regular *if*

(D1) *for all* $n < \omega$*,* $D(n) = D(n) + \mathcal{C}_0$ *(i.e.* $D(n)$ *is "closed under finite perturbations").*

(D2) *for all* $n < \omega$, $X(n) = X_n$ *(with* X_n *as defined in (I)).*

(D, X, d) *is* regular *if furthermore*

(D3) $d = \mathcal{C}$

If $\bar{u} = \bar{v} = \emptyset$ then $c^{(M,\bar{u}),(N,\bar{v})}$ is regular.

By some ad hoc construction of normal models of T we can show

PROPOSITION 3.12. *The application* $((M, \bar{u}), (N, \bar{v})) \mapsto c^{(M,\bar{u}),(N,\bar{v})}$ *maps regular pairs of* 0*-extended models* onto *the set of regular configurations.*

This surjectivity will allow us to work only with configurations and in the end go back to models via that correspondence.

(IV) A configuration $c = (D, X, d)$ is *thick* if $\bigcap_{n \in B} D(n) \cap d \neq \emptyset$ for some *infinite* $B \subset \omega$, otherwise it is *thin*. It is *degenerated* if $D(n) \cap d = \emptyset$ for some $n < \omega$.

If $(D, X, d) = c^{(M,\bar{u}),(N,\bar{v})}$, then $M \cong N$ if and only if $\bigcap_{n<\omega} D(n) \neq \emptyset$ and each element in that intersection is a witness for Theorem 2.5. In particular, if $\bar{u} = \bar{v} = \emptyset$ then thinness of (D, X, d) implies $M \not\cong N$.

DEFINITION 3.13. *By induction on* α *we define a configuration* (D, X, d) *to be* α-rich*:*

- (D, X, d) *is* 0*-rich, if* (D, X, d) *is non degenerated.*
- *For* α *a limit ordinal,* (D, X, d) *is* α*-rich if* (D, X, d) *is* β*-rich for all* $\beta < \alpha$.
- (D, X, d) *is* $\alpha = (\beta + 1)$*-rich if for all* $n < \omega$ *there exists* $\xi_n \in D(n) \cap d$ *such that for all* $\zeta \in \mathcal{C}_0 \cap \mathcal{C}^A$ *(*$A \subset \omega$ *is the unique set satisfying* $d = \eta + \mathcal{C}^A$ *for some* $\eta \in \mathcal{C}$*),* $\xi_n + \zeta \in D(n) \cap d$ *and* $(D, X, d \cap [\xi_i + \zeta]_{X(n)})$ *is* β*-rich, where* $[\xi_n + \zeta]_{X(n)} \subset \mathcal{C}$ *denotes the equivalence class of* $\xi_i + \zeta$ *with respect to the relation* $\sigma_1 \sim \sigma_2$ *if and only if* $\sigma_1 - \sigma_2 \in \mathcal{C}^{X(n)}$ *(i.e. we actually have* $[\xi_n + \zeta]_{X(n)} = (\xi_n + \zeta) + \mathcal{C}^{X(n)}$*).*

Then, using regularity of $c^{(M,\bar{u}),(N,\bar{v})}$, we can show

THEOREM 3.14. *For all* n *and* $((M, \bar{u}), (N, \bar{v}))$ *regular, if* $c^{(M,\bar{u}),(N,\bar{v})}$ *is* α*-rich then* $(M, \bar{u}) \equiv_{\omega+\alpha} (N, \bar{v})$.

The proof goes by induction on α, where the Proposition 3.6 provides the starting point.

(V) Finally, we construct inductively regular configurations c^α which are thin and α-rich. The construction is somewhat lengthy to carry out in detail, although the basic idea is simple. We add inductively exactly those elements to the sets $D(n)$ $(n < \omega)$ which guarantee richness and if we do this in a careful way, namely by choosing elements we add as independent from each other as possible, it will turn out that those configurations will never become thick.

The basic ideas are the following:

(1) For each $s \in \omega^{<\omega}$ we define a monomorphism $\hat{h}_s : \mathcal{C} \to \mathcal{C}$ which intuitively speaking "shrinks" a 0-1-sequence defined on ω to one defined on $X_s := X_{s(0)} \cap X_{s(1)} \cap \cdots \cap X_{s(|s|-1)} \subset \omega$ (see (I) for the definition of the sets X_i).
(2) Then we define a family $(\tilde{\xi}_\beta)_{\beta<\omega_1}$ of elements of $\mathcal{C}$ which are *totally independent*, which is a strong version of linear independence.
(3) We introduce objects Ξ that represent the set of $\tilde{\xi}_\beta$ used in the construction and demand Ξ to be *admissible* which guarantees that in our inductive construction of c^α, we never use two times the same $\tilde{\xi}_\beta$.
(4) We define c^α by adding to c^0 "shrinked" versions of the sets $D(n)$ for all preceeding c^γ ($\gamma < \alpha$), using elements from some admissible Ξ to "label" those added sets (in order to store the information of where they come from).
(5) We show that the c^α are thin.
(6) Finally, we can show that the c^α are α-rich.

Details for (1)-(6) in (V):

For (1): DEFINITION 3.15. *Recall that in (I) we had fixed an enumeration* $(y_i)_{i<\omega}$ *of* $[\omega]^{<\omega} \setminus \{\emptyset\}$. *For* $k < \omega$ *we define*

(i) $g_k : \omega \to \omega$ *by*

$$g_k(n) = \begin{cases} n & \text{if } n < k \\ n+1 & \text{if } n \geq k \end{cases}.$$

(ii) $\tilde{g}_k : [\omega]^{<\omega} \setminus \{\emptyset\} \to [\omega]^{<\omega} \setminus \{\emptyset\}$ *by*

$$\tilde{g}_k(\{n_1, n_2, \ldots, n_l\}) = \{g_k(n_1), g_k(n_2), \ldots, g_k(n_l)\} \cup \{k\}.$$

(iii) $h_k : \omega \to \omega$ *by*

$$h_k(n) = m \quad \Leftrightarrow \quad y_m = \tilde{g}_k(y_n).$$

REMARK 3.16. *We have* $\mathrm{im}(g_k) = \omega \setminus \{k\}$, $\mathrm{im}(\tilde{g}_k) = \{y_i | i \in X_k\}$ *and* $\mathrm{im}(h_k) = X_k$. h_k *is characterized by the property*

$$\forall n < \omega \; y_{h_k(n)} = \tilde{g}(y_n).$$

Moreover it is one-one and for all $i, k < \omega$, $h_k[X_i] = X_k \cap X_{g_k(i)}$.

PROPOSITION 3.17. *Let* $f = h_{k_1} \circ h_{k_2} \circ \cdots \circ h_{k_n}$ *and* $f' = h_{k'_1} \circ h_{k'_2} \circ \cdots \circ h_{k'_m}$. *Then* $\mathrm{im}(f) = \mathrm{im}(f')$ *implies* $n = m$ *and* $f = f'$.

This Proposition allows us to define $h_s : \omega \to \omega$ (for $s \in \omega^{<\omega}$) as the unique composition of functions h_k such that $\mathrm{im}(h_k) = X_s := X_{s(0)} \cap X_{s(1)} \cap \cdots \cap X_{s(|s|-1)}$

Those h_s induce homomorphisms $\hat{h}_s : \mathcal{C} \to \mathcal{C}$ defined as

$$\hat{h}_s(\xi)(n) = \begin{cases} \xi(h_s^{-1}(n)) & \text{if } n \in X_s \\ 0 & \text{if } n \in \omega \setminus X_s \end{cases}$$

We also have inverses to those $\hat{h}_s$: set $h_s^*(\xi) = \xi \circ h_s$ (where we view elements of $\mathcal{C}$ as functions from ω to $\{0,1\}$). Then we have for all $\xi \in \mathcal{C}^{X_s}$:

$$\hat{h}_s(h_s^*(\xi)) = h_s^*(\hat{h}_s(\xi)) = \xi$$

For (2): We begin by *extending* $(X_n)_{n<\omega}$ (which had been defined in (I)) to a family $(X_i)_{i<\omega_1}$ which is still independent in the sense that finite intersections of sets X_i and $\omega \setminus X_j$ are non-empty (and thus infinite). Then define $\tilde{\xi}_\alpha$ as the characteristic function of the set $X_{\omega+\alpha}$. We can show that $(\tilde{\xi}_i)_{i<\omega_1}$ is *totally independent* in the sense of the following definition.

Definition 3.18. *For $X \subset \omega$ let $\pi_X : \mathcal{C} \to \mathcal{C}$ be defined by*

$$\pi_X(\xi)(n) = \begin{cases} \xi(n) & \text{if } n \in X \\ 0 & \text{otherwise} \end{cases}$$

A family $(\sigma_\alpha)_{\alpha<\kappa}$ in $\mathcal{C}$ is totally independent *if for each set $X = X_{i_1} \cap \dots \cap X_{i_k} \cap (\omega \setminus X_{j_1}) \cap \dots \cap (\omega \setminus X_{j_l})$ $(k, l < \omega)$, the family $(\pi_X(\sigma_\alpha))_{\alpha<\kappa}$ is $\mathcal{C}_0$-independent (i.e. no non-trivial finite sum belongs to $\mathcal{C}_0$).*

For further reference, we set

$$\Gamma = \{\tilde{\xi}_\alpha | \alpha < \omega_1\}$$

and

$$\mathcal{X} = \bigcup_{s \in [\omega]^{<\omega}} \hat{h}_s[\Gamma]$$

For (3): For countable α we define recursively

$$A_\alpha = \mathcal{C}^\omega \times \prod_{\beta<\alpha} (\mathcal{C} \times A_\beta)^\omega$$

Let $\Xi \in A_\alpha$ and define $\operatorname{im}(\Xi)$ by induction as follows:

* If $\alpha = 0$ then $\Xi \in \mathcal{C}^\omega$ and we let $\operatorname{im}(\Xi) = \{\Xi(i) | i < \omega\}$ (this is really the image of Ξ considered as a function)
* If $\alpha > 0$ and $\Xi = (\Lambda, ((\xi^{\beta,k}, \Xi^{\beta,k})_{k<\omega})_{\beta<\alpha}) \in A_\alpha$, we let

$$\operatorname{im}(\Xi) = \operatorname{im}(\Lambda) \cup \{\xi^{\beta,k} | k < \omega, \beta < \alpha\} \cup \bigcup_{\beta<\alpha} \bigcup_{k<\omega} \operatorname{im}(\Xi^{\beta,k})$$

Next, we define the *admissible* elements of A_α by induction on α :

* $\Xi \in A_0$ is admissible if Ξ is one-one as a function
* If $\alpha > 0$, $\Xi = (\Lambda, ((\xi^{\beta,k}, \Xi^{\beta,k})_{k<\omega})_{\beta<\alpha}) \in A_\alpha$ is admissible if
  * Λ is one-one
  * all $\Xi^{\beta,k}$ are admissible
  * the elements $\xi^{\beta,k}$ $(\beta < \alpha, k < \omega)$ are distinct
  * the sets $\operatorname{im}(\Lambda)$, $\{\xi^{\beta,k} | k < \omega, \beta < \alpha\}$, $\operatorname{im}(\Xi^{\beta,k})$ (for $\beta < \alpha, k < \omega$) are disjoint

Let $\mathrm{Ad}_\alpha \subset A_\alpha$ be the set of admissible elements of A_α.

For (4): We define configurations $c^{\alpha,\Xi} = (D^{\alpha,\Xi}, X^{\alpha,\Xi}, d^{\alpha,\Xi})$ uniformly for all $\Xi \in A_\alpha$ (and in the end we let c^α be any one of the $c^{\alpha,\Xi}$ with $\Xi \in \mathrm{Ad}_\alpha$ and $\mathrm{im}(\Xi) \subset \mathcal{X}$). Since we want our $c^{\alpha,\Xi}$ to be regular, we set $X^{\alpha,\Xi}(n) = X_n$ for all n and $d^{\alpha,\Xi} = \mathcal{C}$. Thus, it suffices to define $D^{\alpha,\Xi}$.

To start with, let $D^{0,\Xi}(n) = \Xi(n) + \mathcal{C}_0 + \mathcal{C}^{X_n}$.

Then, suppose $D^{\beta,\Xi'}$ already defined for all $\beta < \alpha$ and all $\Xi' \in A_\beta$. $\Xi \in A_\alpha$ is of the form

$$\Xi = (\Lambda, ((\xi^{\beta,k}, \Xi^{\beta,k})_{k<\omega})_{\beta<\alpha}) \in A_\alpha$$

(with $\Lambda \in \mathcal{C}^\omega$, $\xi^{\beta,k} \in \mathcal{C}$ and $\Xi^{\beta,k} \in A_\beta$ for all $k < \omega$ and $\beta < \alpha$). We define for $k, i < \omega$ and $\beta < \alpha$:

$$* \ I^{\beta,\Xi}_{k,i} = \begin{cases} \xi^{\beta,k} + \hat{h}_k[D^{(\beta,\Xi^{\beta,k})}(i-1)] & \text{if } k < i \\ \{\xi^{\beta,k}\} & \text{if } k = i \\ \xi^{\beta,k} + \hat{h}_k[D^{(\beta,\Xi^{\beta,k})}(i)] & \text{if } k > i \end{cases}$$

$$* \ I^{\Xi}_i = \bigcup_{\beta<\alpha} \bigcup_{k<\omega} I^{\beta,\Xi}_{k,i}$$

Now we can define

$$D^{(\alpha,\Xi)}(n) = \Lambda(n) + \langle I^{\Xi}_n - \Lambda(n)\rangle + \mathcal{C}_0 + \mathcal{C}^{X_n},$$

where $\langle I^{\Xi}_n - \Lambda(n)\rangle \subset \mathcal{C}$ is the subgroup of $\mathcal{C}$ generated by the set $I^{\Xi}_n - \Lambda(n) \subset \mathcal{C}$

For (5): The key properties of our construction which will guarantee *thinness* are the total independence of the elements we add, and avoiding repetitions by the admissibility assumption. These properties will allow us to identify at which stage of the construction a given element has been added.

To prove thinness of all $c^{\alpha,\Xi}$ with the additional assumptions that $\Xi \in \mathrm{Ad}_\alpha$ and $\mathrm{im}(\Xi) \subset \mathcal{X}$ (it is not difficult to see that if we drop any of these, there *will be* thick configurations $c^{\alpha,\Xi}$), we first need a few technical lemmas:

LEMMA 3.19. *For any $c^{\alpha,\Xi} = (D, X, d)$, $D(i) \subset \langle\Gamma\rangle + \mathcal{C}_0 + \mathcal{C}^{X_i}$ for all $i < \omega$ (where $\langle\Gamma\rangle$ denotes the subgroup of $\mathcal{C}$ generated by Γ)*

LEMMA 3.20. *For any $s \in [\omega]^{<\omega}\setminus\emptyset$, $\bigcap_{i\in s}(\langle\Gamma\rangle + \mathcal{C}_0 + \mathcal{C}^{X_i}) = \langle\Gamma\rangle + \mathcal{C}_0 + \mathcal{C}^{X_s}$*

NOTATION 3.21. *For any subgroup G of $\mathcal{C}$ and $\xi, \zeta \in \mathcal{C}$, we write $\xi \equiv \zeta \pmod G$ for $\xi - \zeta \in G$.*

LEMMA 3.22. *For all $s_1, s_2, \ldots, s_n \in [\omega]^{<\omega} \setminus \{\emptyset\}$ and $\sigma \in \langle\Gamma\rangle + \mathcal{C}_0 + \mathcal{C}^{\bigcup_{i=1}^n X_{s_i}}$, there is a* unique *finite $A \subset \Gamma \setminus \bigcup_{i=1}^n \hat{h}_{s_i}[\Gamma]$ such that $\sigma \equiv \sum A \pmod{\mathcal{C}_0 + \mathcal{C}^{\bigcup_{i=1}^n X_{s_i}}}$.*
Let $\sigma_{s_1,\ldots,s_n}$ denote the sum $\sum A$ of all elements of A and $C_{s_1,\ldots,s_n}(\sigma) = A$.

The map which to $\sigma \in \langle\Gamma\rangle + \mathcal{C}_0 + \mathcal{C}^{\bigcup_{i=1}^{n} X_{s_i}}$ *assigns* $\sigma_{s_1,\dots,s_n}$ *is a* $(\mathcal{C},+)$*-homomorphism.*

Lemma 3.23. *If* $s \in [\omega]^{<\omega} \setminus \{\emptyset\}$, $\sigma \in \langle\Gamma\rangle + \mathcal{C}_0 + \mathcal{C}^{X_s}$, *then* $C_s(\sigma) = \bigcup_{i\in s} C_{\{i\}}(\sigma)$.

Lemma 3.24. *Let* $s \in [\omega]^{<\omega} \setminus \{\emptyset\}$ *and* $\sigma, \sigma' \in \langle\Gamma\rangle + \mathcal{C}_0 + \mathcal{C}^{X_s}$. *Then* $\sigma_s = \sigma'_s$ *if and only if* $\sigma \equiv \sigma' \pmod{\mathcal{C}_0 + \mathcal{C}^{X_s}}$.

Definition 3.25. *Let* $\mathcal{X}' \subset \mathcal{X}$. *For* $\sigma \in \mathcal{X}$ *(i.e.* σ *is of the form* $\hat{h}_s(\tilde{\xi}_\alpha)$*), we define* $\sigma \restriction \mathcal{X}' = \begin{cases} \sigma & \text{if } \sigma \in \mathcal{X}' \\ 0 & \text{if } \sigma \in \mathcal{X} \setminus \mathcal{X}' \end{cases}$

We extend that definition linearly to $\langle\mathcal{X}\rangle$*: if* $\sigma = \sum_{i=1}^{n} \hat{h}_{t_i}(\tilde{\xi}_{\alpha_i})$, *we set*

$$\sigma \restriction \mathcal{X}' = \sum_{i=1}^{n} (\hat{h}_{t_i}(\tilde{\xi}_{\alpha_i}) \restriction \mathcal{X}')$$

i.e. we drop any element not in $\mathcal{X}'$.

We continue with the proof of thinness in the case of $\alpha = 0$:

Proposition 3.26. *For all* $\Xi \in \mathrm{Ad}_0$ *with* $\mathrm{im}(\Xi) \subset \mathcal{X}$, $c^{0,\Xi}$ *is thin.*

Proof. Let us write $c^{0,\Xi} = (D, X, d)$. For $i, j < \omega$, $i \neq j$, suppose that there exists $\sigma \in D(i) \cap D(j)$. By definition of $c^{0,\Xi}$, $\sigma \equiv \xi(i) \pmod{\mathcal{C}_0 + \mathcal{C}^{X_i}}$ and $\sigma \equiv \xi(j) \pmod{\mathcal{C}_0 + \mathcal{C}^{X_j}}$, which implies $\xi(i) \equiv \xi(j) \pmod{\mathcal{C}_0 + \mathcal{C}^{X_i \cup X_j}}$. Ξ is admissible, i.e. one-one, and since $\mathrm{im}(\Xi) \subset \mathcal{X}$, there are $\alpha \neq \beta$ such that $\xi(i) = \tilde{\xi}_\alpha$ and $\xi(j) = \tilde{\xi}_\beta$. Now, $\tilde{\xi}_\alpha \equiv \tilde{\xi}_\beta \pmod{\mathcal{C}_0 + \mathcal{C}^{X_i \cup X_j}}$ contradicts the fact that $\mathcal{X}$ is totally independent (see (2)). This proves that $D(i) \cap d = D(i) \cap \mathcal{C} = D(i)$ $(i < \omega)$ are pairwise disjoint and in particular that $c^{0,\Xi}$ is thin. ⊣

Now we can prove thinness for any α:

Theorem 3.27. *For all* $\alpha < \omega_1$ *and* $\Xi \in \mathrm{Ad}_\alpha$ *with* $\mathrm{im}(\Xi) \subset \mathcal{X}$, $c^{\alpha,\Xi}$ *is thin.*

Proof. We assume towards a contradiction that there is a minimal α such that for some $\Xi \in \mathrm{Ad}_\alpha$ with $\mathrm{im}(\Xi) \subset \mathcal{X}$, $c^{\alpha,\Xi}$ is thick. Proposition 3.26 shows that $\alpha > 0$. Let $\Xi = (\Lambda, ((\xi^{\beta,k}, \Xi^{\beta,k})_{k<\omega})_{\beta<\alpha})$ (with $\Lambda \in \mathcal{X}^\omega$, $\xi^{\beta,k} \in \mathcal{X}$ and $\Xi^{\beta,k} \in \mathrm{Ad}_\beta$ for all $k < \omega$, $\beta < \alpha$).

We write $c^{\alpha,\Xi} = (D, X, d)$ and the supposed thickness of that configuration gives us an infinite $J \subset \omega$ such that $\bigcap_{k\in J} D(k) \neq \emptyset$ (recall that $d = \mathcal{C}$ by definition). Let $\{j_k | k < \omega\}$ be an enumeration of J and fix an element $\sigma \in \bigcap_{k<\omega} D(j_k)$. We set $s_k = \{j_0, j_1, \dots, j_k\}$ for all $k < \omega$. By the Lemmas 3.19 and 3.20, we have $\sigma \in \langle\Gamma\rangle + \mathcal{C}_0 + \mathcal{C}^{X_{s_k}}$ for all $k < \omega$.

By the construction of $c^{\alpha,\Xi}$, there exists for all $j \in J$ some $m_j < \omega$ and elements $\tau_j^i \in I_j^\Xi$ ($i \in \{1, \ldots, m_j\}$) such that $\sigma \equiv \Lambda(j) + \sum_{i=1}^{m_j}(\tau_j^i - \Lambda(j))$ (mod $\mathcal{C}_0 + \mathcal{C}^{X_j}$). Moreover, by the definition of I_j^Ξ, there are $\beta_j^i < \alpha$ and $k_j^i < \omega$ such that $\tau_j^i = \xi^{\beta_j^i, k_j^i} + \tilde{\tau}_j^i$ with

$$\tilde{\tau}_j^i \in \begin{cases} \hat{h}_{k_j^i}[D^{(\beta_j^i, \Xi^{\beta_j^i, k_j^i})}(j-1)] & \text{if } k_j^i < j \\ \{0\} & \text{if } k_j^i = j \\ \hat{h}_{k_j^i}[D^{(\beta_j^i, \Xi^{\beta_j^i, k_j^i})}(j)] & \text{if } k_j^i > j \end{cases}$$

If $j, l \in J$, $j \neq l$, this implies

$$(m_j + 1)\Lambda(j) + \sum_{i=1}^{m_j}(\xi^{\beta_j^i, k_j^i} + \tilde{\tau}_j^i) \equiv \sigma \ (\text{mod } \mathcal{C}_0 + \mathcal{C}^{X_j})$$

and

$$(m_l + 1)\Lambda(l) + \sum_{i=1}^{m_l}(\xi^{\beta_l^i, k_l^i} + \tilde{\tau}_l^i) \equiv \sigma \ (\text{mod } \mathcal{C}_0 + \mathcal{C}^{X_l}),$$

hence (by total independence and since all $\tilde{\tau}_j^i$ and $\tilde{\tau}_l^i$ are "contracted", i.e. lie in some C^{X_k} (using in Definition 3.18 some set X which is disjoint from the "supports" of the $\tilde{\tau}$ and from X_j, X_l)):

$$(m_j + 1)\Lambda(j) + (m_l + 1)\Lambda(l) + \sum_{i=1}^{m_j} \xi^{\beta_j^i, k_j^i} + \sum_{i=1}^{m_l} \xi^{\beta_l^i, k_l^i} = 0$$

The admissibility of Ξ implies $\Lambda(j) \neq \Lambda(l)$ and
$\Lambda(j), \Lambda(l) \notin \{\xi^{\beta_j^1, k_j^1}, \ldots, \xi^{\beta_j^{m_j}, k_j^{m_j}}\} \cup \{\xi^{\beta_l^1, k_l^1}, \ldots, \xi^{\beta_l^{m_l}, k_l^{m_l}}\}$. Since $\mathcal{X}$ is totally independent, we infer

(1) the numbers m_j and m_l are odd and

(2) $\sum_{i=1}^{m_j} \xi^{\beta_j^i, k_j^i} = \sum_{i=1}^{m_l} \xi^{\beta_l^i, k_l^i}$ and this element is non-zero by (1).

Assuming without loss of generality that the $2n_{j_k} - 1$ first i are exactly those satisfying $\beta_{j_k}^i = \beta$ and $k_{j_k}^i = r$ for some $\beta < \alpha$, $r < \omega$, we can thus write for all $k < \omega$,

$$\sigma \equiv \sum_{i=1}^{2n_{j_k}-1} (\xi^{\beta, r} + \tilde{\tau}_{j_k}^i) + \sum_{i=2n_{j_k}}^{m_{j_k}} (\xi^{\beta_{j_k}^i, k_{j_k}^i} + \tilde{\tau}_{j_k}^i) \ (\text{mod } \mathcal{C}_0 + \mathcal{C}^{X_{j_k}})$$

Using Lemma 3.24, we then have

$$\begin{aligned}\sigma_{\{j_k\}} &= (\sum_{i=1}^{2n_{j_k}-1}(\xi^{\beta,r}+\tilde{\tau}^i_{j_k}) + \sum_{i=2n_{j_k}}^{m_{j_k}}(\xi^{\beta^i_{j_k},k^i_{j_k}}+\tilde{\tau}^i_{j_k}))_{\{j_k\}} \\ &= \sum_{i=1}^{2n_{j_k}-1}((\xi^{\beta,r})_{\{j_k\}}+(\tilde{\tau}^i_{j_k})_{\{j_k\}}) + \sum_{i=2n_{j_k}}^{m_{j_k}}((\xi^{\beta^i_{j_k},k^i_{j_k}})_{\{j_k\}}+(\tilde{\tau}^i_{j_k})_{\{j_k\}}) \\ &= \sum_{i=1}^{2n_{j_k}-1}(\xi^{\beta,r}+(\tilde{\tau}^i_{j_k})_{\{j_k\}}) + \sum_{i=2n_{j_k}}^{m_{j_k}}(\xi^{\beta^i_{j_k},k^i_{j_k}}+(\tilde{\tau}^i_{j_k})_{\{j_k\}})\end{aligned}$$

Let $\mathcal{X}' := \mathrm{im}(\Xi^{\beta,r})$. The admissibility of Ξ implies $\sigma_{\{j_k\}} \restriction \mathcal{X}' = \sum_{i=1}^{2n_{j_k}-1}(\tilde{\tau}^i_{j_k})_{\{j_k\}}$.

From now on, we assume that $j_k > r$ (which is true for all but possibly a finite number of k). We set $\rho := \sum_{i=1}^{2n_{j_k}-1}\tilde{\tau}^i_{j_k}$ and observe that $\rho \in \hat{h}_r[D^{(\beta,\Xi^{\beta,r})}(j_k-1)]$ since it is a sum of an *odd* number of elements of $\hat{h}_r[D^{(\beta,\Xi^{\beta,r})}(j_k-1)]$. Now we see that $\rho_{\{j_k\}} \equiv \rho \pmod{\mathcal{C}_0+\mathcal{C}^{X_{j_k}}}$ and $\rho, \rho_{\{j_k\}} \in \mathcal{C}^{X_r}$, which implies $\rho_{\{j_k\}} - \rho \in (\mathcal{C}_0+\mathcal{C}^{X_{j_k}}) \cap \mathcal{C}^{X_r} = \mathcal{C}_0^{X_r} + \mathcal{C}^{X_{\{r,j_k\}}}$ (writing $C_0^{X_r}$ for $C_0 \cap C^{X_r}$) and since there is a countable group $G \subset \mathcal{C}$ such that $\hat{h}_r[D^{(\beta,\Xi^{\beta,r})}(j_k-1)] = \hat{h}_r[G+\mathcal{C}_0+\mathcal{C}^{X_{j_k-1}}] = \hat{h}_r[G]+\mathcal{C}_0^{X_r}+\mathcal{C}^{X_{\{r,j_k\}}}$, we have

$$\begin{aligned}\sigma_{\{j_k\}} \restriction \mathcal{X}' &= \rho_{\{j_k\}} \\ &= \rho + (\rho_{\{j_k\}} - \rho) \\ &\in \hat{h}_r[D^{(\beta,\Xi^{\beta,r})}(j_k-1)] + \mathcal{C}_0^{X_r} + \mathcal{C}^{X_{\{r,j_k\}}} \\ &= (\hat{h}_r[G] + \mathcal{C}_0^{X_r} + \mathcal{C}^{X_{\{r,j_k\}}}) + \mathcal{C}_0^{X_r} + \mathcal{C}^{X_{\{r,j_k\}}} \\ &= \hat{h}_r[G] + \mathcal{C}_0^{X_r} + \mathcal{C}^{X_{\{r,j_k\}}} \\ &= \hat{h}_r[D^{(\beta,\Xi^{\beta,r})}(j_k-1)]\end{aligned}$$

For all $k < \omega$, we have $\sigma \equiv \sigma_{s_k} \pmod{\mathcal{C}_0 + \mathcal{C}^{X_{s_k}}}$ and in particular $\sigma \equiv \sigma_{s_k} \pmod{\mathcal{C}_0+\mathcal{C}^{X_{j_k}}}$. Lemma 3.24 implies $\sigma_{\{j_k\}} = (\sigma_{s_k})_{\{j_k\}}$, and therefore

$$\sigma_{\{j_k\}} \restriction \mathcal{X}' = (\sigma_{s_k})_{\{j_k\}} \restriction \mathcal{X}' = (\sigma_{s_k} \restriction \mathcal{X}')_{\{j_k\}} \equiv \sigma_{s_k} \restriction \mathcal{X}' \pmod{\mathcal{C}^{X_{j_k}}},$$

i.e. $\nu := \sigma_{s_k} \restriction \mathcal{X}' - \sigma_{\{j_k\}} \restriction \mathcal{X}' \in \mathcal{C}^{X_{j_k}}$. We have already seen that $\rho_{\{j_l\}} \in \mathcal{C}^{X_r}$ for all $l < \omega$, i.e. $C_{\{j_l\}}(\rho) \subset \hat{h}_r[\Gamma]$. The Lemma 3.23 implies $C_{s_k}(\rho) = \bigcup_{i=0}^{k} C_{\{j_i\}}(\rho) \subset \hat{h}_r[\Gamma]$, hence $\rho_{s_k} \in \mathcal{C}^{X_r}$ and also $\sigma_{s_k} \restriction \mathcal{X}' = \rho_{s_k} \in \mathcal{C}^{X_r}$. Consequently, $\nu = \sigma_{s_k} \restriction \mathcal{X}' - \sigma_{\{j_k\}} \restriction \mathcal{X}' \in \mathcal{C}^{X_{j_k}} \cap \mathcal{C}^{X_r} = \mathcal{C}^{X_{\{j_k,r\}}}$.

Since $\sigma_{\{j_k\}} \restriction \mathcal{X}' \in \hat{h}_r[D^{(\beta,\Xi^{\beta,r})}(j_k-1)]$ as we have seen before, there is some $\nu' \in D^{(\beta,\Xi^{\beta,r})}(j_k - 1)$ such that $\sigma_{\{j_k\}} \restriction \mathcal{X}' = \hat{h}_r(\nu')$. Now, $h_r^*[\mathcal{C}^{X_{\{j_k,r\}}}] = \mathcal{C}^{h_r^{-1}[X_{\{j_k,r\}}]} = \mathcal{C}^{X_{j_k-1}}$ (using $h_r[X_{j_k-1}] = X_r \cap X_{g_r(j_k-1)} = X_r \cap X_{j_k}$), and therefore we get

$$\begin{aligned} h_r^*(\sigma_{s_k} \restriction \mathcal{X}') &= h_r^*(\sigma_{\{j_k\}} \restriction \mathcal{X}' + \nu) \\ &= h_r^*(\sigma_{\{j_k\}} \restriction \mathcal{X}') + h_r^*(\nu) \\ &= h_r^*(\hat{h}_r(\nu')) + h_r^*(\nu) \\ &= \nu' + h_r^*(\nu) \\ &\in D^{(\beta,\Xi^{\beta,r})}(j_k - 1) + h_r^*[\mathcal{C}^{X_{\{j_k,r\}}}] \\ &= D^{(\beta,\Xi^{\beta,r})}(j_k - 1) + \mathcal{C}^{X_{j_k-1}} \\ &= D^{(\beta,\Xi^{\beta,r})}(j_k - 1), \end{aligned}$$

which implies

$$\sigma_{s_k} \restriction \mathcal{X}' = \hat{h}_r(h_r^*(\sigma_{s_k} \restriction \mathcal{X}')) \in \hat{h}_r[D^{(\beta,\Xi^{\beta,r})}(j_k - 1)]$$

We observe that the sequence $(\sigma_{s_k} \restriction \mathcal{X}')_{k<\omega}$ converges in $\mathcal{C}$. Indeed, if $l > k$, then $\sigma_{s_l} - \sigma_{s_k} \in \mathcal{C}^{X_{s_k}}$ and moreover, by the definition of the sets X_k, the sequence $(\min(X_{s_r}))_{r<\omega}$ is cofinal in ω. This implies that for all $N < \omega$, there is some N' such that $\forall l > N'$ $\sigma_{s_l} \restriction N = \sigma_{s_{N'}} \restriction N$, meaning that $(\sigma_{s_k})_{k<\omega}$ converges. Now, the same is true for $(\sigma_{s_k} \restriction \mathcal{X}')_{k<\omega}$, since $\sigma_{s_l} - \sigma_{s_k} \in \mathcal{C}^{X_{s_k}}$ implies $\sigma_{s_l} \restriction \mathcal{X}' - \sigma_{s_k} \restriction \mathcal{X}' \in \mathcal{C}^{X_{s_k}}$.

Let $\sigma_{\beta,r} := \lim_{k\to\infty}(\sigma_{s_k} \restriction \mathcal{X}')$. We fix $k < \omega$ and use the fact that $\sigma_{s_l} \restriction \mathcal{X}' \equiv \sigma_{s_k} \restriction \mathcal{X}' \pmod{\mathcal{C}_0 + \mathcal{C}^{X_{s_k}}}$ for all $l \geq k$, which in particular implies $\sigma_{s_l} \restriction \mathcal{X}' \equiv \sigma_{s_k} \restriction \mathcal{X}' \pmod{\mathcal{C}_0 + \mathcal{C}^{X_{j_k}}}$. Since $\sigma_{s_l} \restriction \mathcal{X}', \sigma_{s_k} \restriction \mathcal{X}' \in \mathcal{C}^{X_r}$, we have $\sigma_{s_l} \restriction \mathcal{X}' - \sigma_{s_k} \restriction \mathcal{X}' \in \mathcal{C}^{X_{\{j_k,r\}}}$ for all $l \geq k$. Now, $\mathcal{C}^{X_{\{j_k,r\}}}$ is a closed set and thus,

$$\sigma_{\beta,r} - (\sigma_{s_k} \restriction \mathcal{X}') = \lim_{l\to\infty}(\sigma_{s_l} \restriction \mathcal{X}') - \sigma_{s_k} \restriction \mathcal{X}' = \lim_{l\to\infty}(\sigma_{s_l} \restriction \mathcal{X}' - \sigma_{s_k} \restriction \mathcal{X}') \in \mathcal{C}^{X_{\{j_k,r\}}}.$$

We define $\mu = \sigma_{\beta,r} - (\sigma_{s_k} \restriction \mathcal{X}')$ (which belongs to $\mathcal{C}^{X_{\{j_k,r\}}}$). We have proved that $\sigma_{s_k} \restriction \mathcal{X}' \in \hat{h}_r[D^{(\beta,\Xi^{\beta,r})}(j_k - 1)]$, therefore there exists some $\mu' \in D^{(\beta,\Xi^{\beta,r})}(j_k - 1)$ such that $\sigma_{s_k} \restriction \mathcal{X}' = \hat{h}_r(\mu')$. It follows that

$$\begin{aligned} h_r^*(\sigma_{\beta,r}) &= h_r^*(\sigma_{s_k} \restriction \mathcal{X}' + \mu) \\ &= h_r^*(\sigma_{s_k} \restriction \mathcal{X}') + h_r^*(\mu) \\ &= h_r^*(\hat{h}_r(\mu')) + h_r^*(\mu) \\ &= \mu' + h_r^*(\mu) \\ &\in D^{(\beta,\Xi^{\beta,r})}(j_k - 1) + h_r^*[\mathcal{C}^{X_{\{j_k,r\}}}] \\ &= D^{(\beta,\Xi^{\beta,r})}(j_k - 1) + \mathcal{C}^{X_{j_k-1}} \\ &= D^{(\beta,\Xi^{\beta,r})}(j_k - 1), \end{aligned}$$

We have shown that $h_r^*(\sigma_{\beta,r}) \in D^{(\beta,\Xi^{\beta,r})}(j_k - 1)$ for all $k < \omega$, which implies

$\bigcap_{k<\omega} D^{(\beta,\Xi^{\beta,r})}(j_k-1) \neq \emptyset$. This means that $c^{\beta,\Xi^{\beta,r}}$ is thick and contradicts the minimal choice of α. ⊣

For (6): We essentially added at each step of the inductive construction of the $c^{\alpha,\Xi}$ to the sets $D(i)$ what was necessary to raise the richness by one step. The formal proof of α-richness for $c^{\alpha,\Xi} = (D, X, d)$ goes by induction. We have to introduce several ways of manipulating configurations, to be able to eventually apply the induction hypothesis. That is where we need the extra information provided by X and d and where we need to modify them.

Recall that in the definition of α-richness we have applied the following kind of modification to a configuration:

Definition 3.28. *If $c = (D, X, d)$ is a configuration and $Z = \xi + \mathcal{C}^A$ ($A \subset \omega$, $\xi \in \mathcal{C}$), we define $c^\frown Z = (D, X, c \cap Z)$*

Now follows a very similar manipulation, only that D and X are also "cut by Z":

Definition 3.29. *If $c = (D, X, d)$ is a configuration and $Z = \xi + \mathcal{C}^A$ ($A \subset \omega$, $\xi \in \mathcal{C}$), we define $c \cap Z = (\bar{D}, \bar{X}, \bar{d})$ by*
- $\bar{D}(i) = D(i) \cap Z$
- $\bar{X}(i) = X(i) \cap A$
- $\bar{d} = d \cap Z$

(for all $i < \omega$)

Then we can show by induction:

Lemma 3.30. *Let $c = (D, X, d)$ be a configuration, $Z = \xi + \mathcal{C}^A$ ($A \subset \omega$, $\xi \in \mathcal{C}$), and let $\alpha < \omega_1$. Then $c^\frown Z$ is α-rich if and only if $c \cap Z$ is α-rich.*

For the following definition, remember that for a one-one function $h : \omega \to \omega$, $h^* : \mathcal{C} \to \mathcal{C}$ had been defined as $h^*(\xi) = \xi \circ h$.

Definition 3.31. *Let $h : \omega \to \omega$ be one-one and $c = (D, X, d)$ a configuration. We define $h^*(c) = (\bar{D}, \bar{X}, \bar{d})$ by*
- $\bar{D}(i) = h^*[D(i)]$
- $\bar{X}(i) = h^{-1}[X(i)]$
- $\bar{d} = h^*[d]$

(for all $i < \omega$).

Using lemma 3.30, we can inductively show:

Lemma 3.32. *Let $h : \omega \to A$ be bijective, $c = (D, X, d)$ a configuration and $Z = \rho + \mathcal{C}^A$ ($\rho \in \mathcal{C}$). For all α, if $h^*(c \cap Z)$ is α-rich, $c^\frown Z$ is α-rich as well.*

Next, we define a shift and a translation operation for configurations:

Definition 3.33. *Let $c = (D, X, d)$ be a configuration, $k < \omega$ and $\xi \in \mathcal{C}$. We define*
- $\mathrm{dec}_k(c) = (\bar{D}, \bar{X}, \bar{d})$ *by*

· $\bar{D}(i) = \begin{cases} D(i) & \textit{if } i < k \\ \mathcal{C} & \textit{if } i = k \\ D(i-1) & \textit{if } i > k \end{cases}$

· $\bar{X}(i) = \begin{cases} X(i) & \textit{if } i < k \\ \omega & \textit{if } i = k \\ X(i-1) & \textit{if } i > k \end{cases}$

· $\bar{d} = d$

(for all $i < \omega$).

∗ $\xi + c = (\bar{D}, \bar{X}, \bar{d})$ *by*

· $\bar{D}(i) = \xi + D(i)$

· $\bar{X}(i) = X(i)$

· $\bar{d} = \xi + d$

(for all $i < \omega$).

and are able to prove

LEMMA 3.34. *If $c = (D, X, d)$ is α-rich,* $\mathrm{dec}_k(c)$ *and $\xi + c$ are still α-rich, (for all $k < \omega$ and $\xi \in \mathcal{C}$).*

Finally, we have a subset relation for configurations:

DEFINITION 3.35. *Let $c = (D, X, d)$, $c' = (D', X', d')$ be configurations. We define the relation $c \subset c'$ by*

∗ $D(i) \subset D'(i)$

∗ $X = X'$

∗ $d = d'$

for all $i < \omega$.

and not surprisingly

LEMMA 3.36. *If c is α-rich and $c \subset c'$, then c' is also α-rich.*

With these preparations we can proceed to the proof of α-richness:

THEOREM 3.37. *For all $\alpha < \omega_1$ and $\Xi \in A_\alpha$, $c^{\alpha,\Xi}$ is α-rich.*

PROOF. By induction on α (for all Ξ simultaneously).

By construction, $c^{0,\Xi}$ is non-degenerated.

Let $\alpha > 0$ and $\Xi = (\Lambda, ((\xi^{\beta,k}, \Xi^{\beta,k})_{k<\omega})_{\beta<\alpha})$ (with $\Lambda \in \mathcal{C}^\omega$, $\xi^{\beta,k} \in \mathcal{C}$, $\Xi^{\beta,k} \in A_\beta$ for all $k < \omega$, $\beta < \alpha$).
We fix some $\gamma < \alpha$ and show that $c^{\alpha,\Xi} = (D, X, d)$ is $(\gamma+1)$-rich. We verify that $\xi^{\gamma,i}$ witnesses this for i, i.e.

$$\forall \zeta \in \mathcal{C}_0 (\xi^{\gamma,i} + \zeta \in D(i) \text{ and } (c^{\alpha,\Xi})^\frown[\xi^{\gamma,i} + \zeta]_{X(i)} \text{ is } \gamma\text{-rich})$$

Fix some $\zeta \in \mathcal{C}_0$. First, $\xi^{\gamma,i} + \zeta \in D(i)$ since by construction, $\xi^{\gamma,i} \in D(i)$ and $D(i) = D(i) + \mathcal{C}_0$ ($c^{\alpha,\Xi}$ is regular).
By the Lemma 3.32, it is enough to show that $h_i^*((c^{\alpha,\Xi}) \cap [\xi^{\gamma,i} + \zeta]_{X(i)})$ is γ-rich. We will show that for $\rho := h_i^*(\xi^{\gamma,i} + \zeta)$,

$$\mathrm{dec}_i(\rho + c^{\gamma,\Xi^{\gamma,i}}) \subset h_i^*((c^{\alpha,\Xi}) \cap [\xi^{\gamma,i} + \zeta]_{X(i)}),$$

which finishes the proof by the Lemmas 3.34 and 3.36, since $c^{\gamma,\Xi^{\gamma,i}}$ is γ-rich by induction.

We write $\mathrm{dec}_i(\rho + c^{\gamma,\Xi^{\gamma,i}}) = (\bar{D}, \bar{X}, \bar{d})$ and $h_i^*((c^{\alpha,\Xi}) \cap [\xi^{\gamma,i} + \zeta]_{X(i)}) = (D', X', d')$ and verify that for all $j < \omega$, $\bar{D}(j) \subset D'(j)$, $\bar{X}(j) = X'(j)$ and $\bar{d} = d'$. There are three cases:

(1) Suppose $j < i$. First we see that by definition,
- $\bar{D}(j) = \rho + D^{(\gamma,\Xi^{\gamma,i})}(j)$
- $\bar{X}(j) = X^{(\gamma,\Xi^{\gamma,i})}(j) = X(j)$
- $\bar{d} = \rho + d^{(\gamma,\Xi^{\gamma,i})} = \rho + \mathcal{C} = \mathcal{C}$

Then,

$$\begin{aligned}
D'(j) &= h_i^*[D(j) \cap [\xi^{\gamma,i} + \zeta]_{X(i)}] \\
&= h_i^*[(\Lambda(j) + \langle I_j^{\Xi} - \Lambda(j)\rangle + \mathcal{C}_0 + \mathcal{C}^{X_j}) \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \\
&\supset h_i^*[(\Lambda(j) + (I_{i,j}^{\gamma,\Xi} - \Lambda(j)) + \zeta + \mathcal{C}^{X_j}) \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \\
&= h_i^*[(I_{i,j}^{\gamma,\Xi} + \zeta + \mathcal{C}^{X_j}) \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \\
&= h_i^*[(\xi^{\gamma,i} + \hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j)] + \zeta + \mathcal{C}^{X_j}) \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \\
&= h_i^*[(\xi^{\gamma,i} + \zeta) + ((\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j)] + \mathcal{C}^{X_j}) \cap \mathcal{C}^{X_i})] \\
&= h_i^*(\xi^{\gamma,i} + \zeta) + h_i^*[(\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j)] + \mathcal{C}^{X_j}) \cap \mathcal{C}^{X_i}] \\
&\supset h_i^*(\xi^{\gamma,i} + \zeta) + h_i^*[(\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j)] + \mathcal{C}^{X_j \cap X_i}) \cap \mathcal{C}^{X_i}] \\
&= \rho + h_i^*[\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j)] + \mathcal{C}^{X_j \cap X_i}] \\
&\supset \rho + h_i^*[\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j)]] \\
&= \rho + D^{(\gamma,\Xi^{\gamma,i})}(j) \\
&= \bar{D}(j)
\end{aligned}$$

Also,

- $X'(j) = h_i^{-1}[X(j) \cap X_i] = h_i^{-1}[X_j \cap X_i] = X_j = \bar{X}(j)$ (because $h_i[X_j] = X_i \cap X_{g_i(j)} = X_i \cap X_j$ since $j < i$)
- $d' = h_i^*[d \cap [\xi^{\gamma,i} + \zeta]_{X(i)}] = h_i^*[\mathcal{C} \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \supset h_i^*[\mathcal{C}^{X_i}] = \mathcal{C}^{h_i^{-1}[X_i]} = \mathcal{C} = \bar{d}$

(2) Now, let $j > i$. We have
- $\bar{D}(j) = \rho + D^{(\gamma,\Xi^{\gamma,i})}(j-1)$
- $\bar{X}(j) = X^{(\gamma,\Xi^{\gamma,i})}(j-1) = X(j-1)$
- $\bar{d} = \rho + d^{(\gamma,\Xi^{\gamma,i})} = \rho + \mathcal{C} = \mathcal{C}$

Then,

$$\begin{aligned}
D'(j) &= h_i^*[D(j) \cap [\xi^{\gamma,i} + \zeta]_{X(i)}] \\
&= h_i^*[(\Lambda(j) + \langle I_j^{\Xi} - \Lambda(j)\rangle + \mathcal{C}_0 + \mathcal{C}^{X_j}) \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \\
&\supset h_i^*[(\Lambda(j) + (I_{i,j}^{\gamma,\Xi} - \Lambda(j)) + \zeta + \mathcal{C}^{X_j}) \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \\
&= h_i^*[(I_{i,j}^{\gamma,\Xi} + \zeta + \mathcal{C}^{X_j}) \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \\
&= h_i^*[(\xi^{\gamma,i} + \hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j-1)] + \zeta + \mathcal{C}^{X_j}) \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \\
&= h_i^*[(\xi^{\gamma,i} + \zeta) + ((\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j-1)] + \mathcal{C}^{X_j}) \cap \mathcal{C}^{X_i})] \\
&= h_i^*(\xi^{\gamma,i} + \zeta) + h_i^*[(\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j-1)] + \mathcal{C}^{X_j}) \cap \mathcal{C}^{X_i}] \\
&\supset h_i^*(\xi^{\gamma,i} + \zeta) + h_i^*[(\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j-1)] + \mathcal{C}^{X_j \cap X_i}) \cap \mathcal{C}^{X_i}] \\
&= \rho + h_i^*[\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j-1)] + \mathcal{C}^{X_j \cap X_i}] \\
&\supset \rho + h_i^*[\hat{h}_i[D^{(\gamma,\Xi^{\gamma,i})}(j-1)]] \\
&= \rho + D^{(\gamma,\Xi^{\gamma,i})}(j-1) \\
&= \bar{D}(j)
\end{aligned}$$

- $X'(j) = h_i^{-1}[X(j) \cap X_i] = h_i^{-1}[X_j \cap X_i] = X_{j-1} = \bar{X}(j)$ (because $h_i[X_{j-1}] = X_i \cap X_{g_i(j-1)} = X_i \cap X_j$ since $j < i$)
- $d' = h_i^*[d \cap [\xi^{\gamma,i} + \zeta]_{X(i)}] = h_i^*[\mathcal{C} \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \supset h_i^*[\mathcal{C}^{X_i}] = \mathcal{C}^{h_i^{-1}[X_i]} = \mathcal{C} = \bar{d}$

(3) Let finally $j = i$. By definition,
- $\bar{D}(j) = \mathcal{C}$
- $\bar{X}(j) = \omega$
- $\bar{d} = \rho + d^{(\gamma,\Xi^{\gamma,i})} = \rho + \mathcal{C} = \mathcal{C}$

On the other hand,

$$\begin{aligned}
D'(j) &= h_i^*[D(j) \cap [\xi^{\gamma,i} + \zeta]_{X(i)}] \\
&= h_i^*[(\Lambda(j) + \langle I_j^{\Xi} - \Lambda(j)\rangle + \mathcal{C}_0 + \mathcal{C}^{X_i}) \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i})] \\
&\supset h_i^*[\mathcal{C}^{X_i} \cap \mathcal{C}^{X_i}] \\
&= h_i^*[\mathcal{C}^{X_i}] \\
&= \mathcal{C} \\
&= \bar{D}(j)
\end{aligned}$$

- $X'(j) = h_i^{-1}[X(j) \cap X_i] = h_i^{-1}[X_j \cap X_i] = h_i^{-1}[X_i \cap X_i] = h_i^{-1}[X_i] = \omega = \bar{X}(j)$
- $d' = h_i^*[d \cap [\xi^{\gamma,i} + \zeta]_{X(i)}] = h_i^*[\mathcal{C} \cap (\xi^{\gamma,i} + \zeta + \mathcal{C}^{X_i}] \supset h_i^*[\mathcal{C}^{X_i}] = \mathcal{C}^{h_i^{-1}[X_i]} = \mathcal{C} = \bar{d}$

⊣

Symbol	Page	Part
a_i^j	3	last paragraph
$\mathcal{A}$	4	Definition 2.3
δ_b^M	4	Definition 2.3
$\mathcal{C}$	4	Definition 2.4
$\equiv_\alpha$	4	Definition 3.1
$\mathrm{SH}(M)$, $\mathrm{SH}(T)$	5	Definition 3.1
$\mathrm{Stab}(\delta)$	5	First paragraph of (I)
$\mathcal{C}\delta$	5	First paragraph of (I)
$\mu(\delta, M)$	5	Second paragraph of (I)
M^o	5	Definition 3.3
$S^{M,o}$	6	Definition 3.4
y_i	6	Third paragraph after Definition 3.4
X_i	6	Third paragraph after Definition 3.4
δ_i, C^X	6	Before Definition 3.5
$\mathcal{C}_0$	6	Definition 3.5
$\Delta^{(M,N),o}$	6	First paragraph of (II)
$K^{(M,\bar{u}),(N,\bar{v})}$	6	Definition 3.7
$o[u]$	7	Definition 3.8
$M^{\bar{u}}$	7	Definition 3.8
$\mathcal{P}(A)$	7	First paragraph of (III)
$c^{(M,\bar{u}),(N,\bar{v})}$	7	First paragraph after Definition 3.10
$[\xi]_X$	8	Definition 3.13
g_k	9	Definition 3.15(i)
$\tilde{g}_k$	9	Definition 3.15(ii)
h_k	9	Definition 3.15(iii)
h_s	9	First paragraph after Proposition 3.17
$\hat{h}_s$	9	Second paragraph after Proposition 3.17
h_s^*	10	First paragraph
π_X	10	Definition 3.18
Γ, $\mathcal{X}$	10	First paragraph after Definition 3.18
A_α	10	First paragraph of "For (3)"
$\mathrm{im}(\Xi)$	10	Second paragraph of "For (3)"
Ad_α	11	First paragraph
$c^{\alpha,\Xi}$, $D^{\alpha,\Xi}$, I_i^Ξ, $\langle A\rangle$	11	"For (4)"
$\xi \equiv \zeta \pmod G$	11	Notation 3.21
$\sigma_{(s_1,\ldots,s_n)}$, $C_{(s_1,\ldots,s_n)}(\sigma)$	11	Last paragraph
$\mathcal{C}_0^X$	14	Third paragraph
$c^\frown Z$	16	Definition 3.28
$c \cap Z$	16	Definition 3.29
$h^*(c)$	16	Definition 3.31
$mathrmdec_k(c)$	16	Definition 3.33
$\xi + c$	17	Definition 3.33
$c \subset c'$	17	Definition 3.35

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